Perturbation Theory of Dynamical Systems

Nils Berglund
Department of Mathematics
ETH Zürich
8092 **Zürich**
Switzerland

Lecture Notes
Summer Semester 2001

Version:
November 14, 2001

Preface

This text is a slightly edited version of lecture notes for a course I gave at ETH, during the Summer term 2001, to undergraduate Mathematics and Physics students. It covers a few selected topics from perturbation theory at an introductory level. Only certain results are proved, and for some of the most important theorems, sketches of the proofs are provided.

Chapter 2 presents a topological approach to perturbations of planar vector fields. It is based on the idea that the qualitative dynamics of most vector fields does not change under small perturbations, and indeed, the set of all these structurally stable systems can be identified. The most common exceptional vector fields can be classified as well.

In Chapter 3, we use the problem of stability of elliptic periodic orbits to develop perturbation theory for a class of dynamical systems of dimension 3 and larger, including (but not limited to) integrable Hamiltonian systems. This will bring us, via averaging and Lie-Deprit series, all the way to KAM-theory.

Finally, Chapter 4 contains an introduction to singular perturbation theory, which is concerned with systems that do not admit a well-defined limit when the perturbation parameter goes to zero. After stating a fundamental result on existence of invariant manifolds, we discuss a few examples of dynamic bifurcations.

An embarrassing number of typos from the first version has been corrected, thanks to my students' attentiveness.

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Please send any comments to `berglund@math.ethz.ch`

Zürich, November 2001

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Chapter 1

Introduction and Examples

The main purpose of perturbation theory is easy to describe. Consider for instance an ordinary differential equation of the form

$$\dot{x} = f(x, \varepsilon), \tag{1.0.1}$$

where $x \in \mathbb{R}^n$ defines the state of the system, $\varepsilon \in \mathbb{R}$ is a parameter, $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a sufficiently smooth vector field, and the dot denotes derivation with respect to time. Assume that the dynamics of the unperturbed equation

$$\dot{x} = f(x, 0) \tag{1.0.2}$$

is well understood, meaning either that we can solve this system exactly, or, at least, that we have a good knowledge of its long-time behaviour. What can be said about solutions of (1.0.1), especially for $t \to \infty$, if ε is sufficiently small?

The answer to this question obviously depends on the dynamics of (1.0.2), as well as on the kind of ε-dependence of f. There are several reasons to be interested in this problem:

- Only very few differential equations can be solved exactly, and if we are able to describe the dynamics of a certain class of perturbations, we will significantly increase the number of systems that we understand.
- In applications, many systems are described by differential equations, but often with an imprecise knowledge of the function f; it is thus important to understand the stability of the dynamics under small perturbations of f.
- If the dynamics of (1.0.1) is qualitatively the same for some set of values of ε, we can construct equivalence classes of dynamical systems, and it is sufficient to study one member of each class.

Many different methods have been developed to study perturbation problems of the form (1.0.1), and we will present a selection of them. Before doing that for general systems, let us discuss a few examples.

1.1 One-Dimensional Examples

One-dimensional ordinary differential equations are the easiest to study, and in fact, their solutions never display a very interesting behaviour. They provide, however, good examples to illustrate some basic facts of perturbation theory.

Example 1.1.1. Consider the equation

$$\dot{x} = f(x,\varepsilon) = -x + \varepsilon g(x). \tag{1.1.1}$$

When $\varepsilon = 0$, the solution with initial condition $x(0) = x_0$ is

$$x(t) = x_0 \, \mathrm{e}^{-t}, \tag{1.1.2}$$

and thus $x(t)$ converges to 0 as $t \to \infty$ for any initial condition x_0. Our aim is to find a class of functions g such that the solutions of (1.1.1) behave qualitatively the same for sufficiently small ε. We will compare different methods to achieve this.

1. **Method of exact solution:** The solution of (1.1.1) with initial condition $x(0) = x_0$ satisfies

$$\int_{x_0}^{x(t)} \frac{\mathrm{d}x}{-x + \varepsilon g(x)} = t. \tag{1.1.3}$$

   Thus the equation can be solved in principle by computing the integral and solving with respect to $x(t)$. Consider the particular case $g(x) = x^2$. Then the integral can be computed by elementary methods, and the result is

$$x(t) = \frac{x_0 \, \mathrm{e}^{-t}}{1 - \varepsilon x_0 (1 - \mathrm{e}^{-t})}. \tag{1.1.4}$$

   Analysing the limit $t \to \infty$, we find that

   - if $x_0 < 1/\varepsilon$, then $x(t) \to 0$ as $t \to \infty$;
   - if $x_0 = 1/\varepsilon$, then $x(t) = x_0$ for all t;
   - if $x_0 > 1/\varepsilon$, then $x(t)$ diverges for a time $t^\star$ given by $\mathrm{e}^{-t^\star} = 1 - 1/(\varepsilon x_0)$.

   This particular case shows that the perturbation term εx^2 indeed has a small influence as long as the initial condition x_0 is not too large (and the smaller ε, the larger the initial condition may be). The obvious advantage of an exact solution is that the long-term behaviour can be analysed exactly. There are, however, important disadvantages:

   - for most functions $g(x)$, the integral in (1.1.3) cannot be computed exactly, and it is thus difficult to make statements about general classes of functions g;
   - even if we manage to compute the integral, quite a bit of analysis may be required to extract information on the asymptotic behaviour.

   Of course, even if we cannot compute the integral, we may still be able to extract some information on the limit $t \to \infty$ directly from (1.1.3). In more than one dimension, however, formulae such as (1.1.3) are not available in general, and thus alternative methods are needed.

2. **Method of series expansion:** A second, very popular method, is to look for solutions of the form

$$x(t) = x^0(t) + \varepsilon x^1(t) + \varepsilon^2 x^2(t) + \dots \tag{1.1.5}$$

   with initial conditions $x^0(0) = x_0$ and $x^j(0) = 0$ for $j \geqslant 1$. By plugging this expression into (1.1.1), and assuming that g is sufficiently differentiable that we can write

$$\varepsilon g(x) = \varepsilon g(x^0) + \varepsilon^2 g'(x_0) x^1 + \dots \tag{1.1.6}$$

we arrive at a hierarchy of equations that can be solved recursively:

$$\begin{aligned}
\dot{x}^0 &= -x^0 && \Rightarrow && x^0(t) = x_0\,\mathrm{e}^{-t} \\
\dot{x}^1 &= -x^1 + g(x^0) && \Rightarrow && x^1(t) = \int_0^t \mathrm{e}^{-(t-s)}\, g(x^0(s))\,\mathrm{d}s \\
\dot{x}^2 &= -x^2 + g'(x^0)x^1 && \Rightarrow && x^2(t) = \int_0^t \mathrm{e}^{-(t-s)}\, g'(x^0(s))x^1(s)\,\mathrm{d}s \\
& && \dots
\end{aligned} \tag{1.1.7}$$

The advantage of this method is that the computations are relatively easy and systematic. Indeed, at each order we only need to solve an inhomogeneous linear equation, where the inhomogeneous term depends only on previously computed quantities.

The disadvantages of this method are that it is difficult to control the convergence of the series (1.1.5), and that it depends on the smoothness of g (if g has a finite number of continuous derivatives, the expansion (1.1.5) has to be finite, with the last term depending on ε). If g is analytic, it is possible to investigate the convergence of the series, but this requires quite heavy combinatorial calculations.

3. **Iterative method:** This variant of the previous method solves some of its disadvantages. Equation (1.1.1) can be treated by the method of variation of the constant: putting $x(t) = c(t)\,\mathrm{e}^{-t}$, we get an equation for $c(t)$ leading to the relation

$$x(t) = x_0\,\mathrm{e}^{-t} + \varepsilon \int_0^t \mathrm{e}^{-(t-s)}\, g(x(s))\,\mathrm{d}s. \tag{1.1.8}$$

One can try to solve this equation by iterations, setting $x^{(0)}(t) = x_0\,\mathrm{e}^{-t}$ and defining a sequence of functions recursively by $x^{(n+1)} = \mathcal{T}x^{(n)}$, where

$$(\mathcal{T}x)(t) = x_0\,\mathrm{e}^{-t} + \varepsilon \int_0^t \mathrm{e}^{-(t-s)}\, g(x(s))\,\mathrm{d}s. \tag{1.1.9}$$

If $\mathcal{T}$ is a contraction in an appropriate Banach space, then the sequence $(x^{(n)})_{n\geqslant 0}$ will converge to a solution of (1.1.8). It turns out that if g satisfies a uniform global Lipschitz condition

$$|g(x) - g(y)| \leqslant K|x-y| \qquad \forall x, y \in \mathbb{R}\,, \tag{1.1.10}$$

and for the norm

$$\|x\|_t = \sup_{0\leqslant s\leqslant t} |x(s)|, \tag{1.1.11}$$

we get from the definition of $\mathcal{T}$ the estimate

$$\begin{aligned}
|(\mathcal{T}x)(t) - (\mathcal{T}y)(t)| &\leqslant \varepsilon \int_0^t \mathrm{e}^{-(t-s)}\, K|x(s)-y(s)|\,\mathrm{d}s \\
&\leqslant \varepsilon K(1-\mathrm{e}^{-t})\|x-y\|_t.
\end{aligned} \tag{1.1.12}$$

Thus if $\varepsilon < 1/K$, $\mathcal{T}$ is a contraction with contraction constant $\lambda = \varepsilon K(1-\mathrm{e}^{-t})$. The iterates $x^{(n)}(t)$ converge to the solution $x(t)$, and we can even estimate the rate of convergence:

$$
\begin{aligned}
\|x^{(n)} - x\|_t &\leqslant \sum_{m=n}^{\infty} \|x^{(m+1)} - x^{(m)}\|_t \\
&\leqslant \sum_{m=n}^{\infty} \lambda^m \|x^{(1)} - x^{(0)}\|_t = \frac{\lambda^n}{1-\lambda} \|x^{(1)} - x^{(0)}\|_t.
\end{aligned}
\tag{1.1.13}
$$

Thus after n iterations, we will have approximated $x(t)$ up to order ε^n.

The advantage of the iterative method is that it allows for a precise control of the convergence, under rather weak assumptions on g (if g satisfies only a local Lipschitz condition, as in the case $g(x) = x^2$, the method can be adapted by restricting the domain of $\mathcal{T}$, which amounts to restricting the domain of initial conditions x_0).

However, this method usually requires more calculations than the previous one, which is why one often uses a combination of both methods: iterations to prove convergence, and expansions to compute the terms of the series. It should also be noted that the fact that $\mathcal{T}$ is contracting or not will depend on the unperturbed vector field $f(x,0)$.

4. **Method of changes of variables:** There exists still another method to solve (1.1.1) for small ε, which is usually more difficult to implement, but often easier to extend to other unperturbed vector fields and higher dimension.

   Let us consider the effect of the change of variables $x = y + \varepsilon h(y)$, where the function h will be chosen in such a way as to make the equation for y as simple as possible. Replacing x by $y + \varepsilon h(y)$ in (1.1.1), we get

$$
\dot y + \varepsilon h'(y)\dot y = -y - \varepsilon h(y) + \varepsilon g(y + \varepsilon h(y)).
\tag{1.1.14}
$$

   Ideally, the transformation should remove the perturbation term $\varepsilon g(x)$. Let us thus impose that

$$
\dot y = -y.
\tag{1.1.15}
$$

   This amounts to requiring that

$$
h'(y) = \frac{1}{y} h(y) - \frac{1}{y} g(y + \varepsilon h(y)).
\tag{1.1.16}
$$

   If such a function h can be found, then the solution of (1.1.1) is determined by the relations

$$
\begin{aligned}
x(t) &= y(t) + \varepsilon h(y(t)) = y_0 \,\mathrm{e}^{-t} + \varepsilon h(y_0 \,\mathrm{e}^{-t}) \\
x_0 &= y_0 + \varepsilon h(y_0).
\end{aligned}
\tag{1.1.17}
$$

   In particular, we would have that $x(t) \to \varepsilon h(0)$ as $t \to \infty$. We can try to construct a solution of (1.1.16) by the method of variation of the constant. Without g, $h(y) = y$ is a solution. Writing $h(y) = c(y)y$ and solving for $c(y)$, we arrive at the relation

$$
h(y) = -y \int_a^y \frac{1}{z^2} g(z + \varepsilon h(z)) \,\mathrm{d}z =: (\mathcal{T}h)(y),
\tag{1.1.18}
$$

where a is an arbitrary constant. It turns out that if g satisfies the Lipschitz condition (1.1.10) and $|a| \to \infty$, then $\mathcal{T}$ is a contraction if $\varepsilon < 1/K$, and thus h exists and can be computed by iterations.

For more general $f(x, 0)$ it is usually more difficult to find h, and one has to construct it by successive approximations.

5. **Geometrical method:** The geometrical approach gives less detailed informations than the previous methods, but can be extended to a larger class of equations and requires very little computations.

   In dimension one, the basic observation is that $x(t)$ is (strictly) increasing if and only if $f(x(t), \varepsilon)$ is (strictly) positive. Thus $x(t)$ either converges to a zero of f, or it is unbounded. Hence the asymptotic behaviour of $x(t)$ is essentially governed by the zeroes and the sign of f. Now we observe that if g satisfies the Lipschitz condition (1.1.10), then for all $y > x$ we have

$$\begin{aligned} f(y, \varepsilon) - f(x, \varepsilon) &= -y + x + \varepsilon[g(y) - g(x)] \\ &\leqslant -(1 - \varepsilon K)(y - x), \end{aligned} \tag{1.1.19}$$

   and thus f is monotonously decreasing if $\varepsilon < 1/K$. Since $f(x, \varepsilon) \to \mp\infty$ as $x \to \pm\infty$, it follows that f has only one singular point, which attracts all solutions.

   If g satisfies only a local Lipschitz condition, as in the case $g(x) = x^2$, one can show that for small ε there exists a point near or at 0 attracting solutions starting in some neighbourhood.

We have just seen various methods showing that the equation $\dot{x} = -x$ is robust under quite a general class of small perturbations, in the sense that solutions of the perturbed system are also attracted by a single point near the origin, at least when they start not to far from 0. The system $\dot{x} = -x$ is called *structurally stable.*

We will now give an example where the situation is not so simple.

Example 1.1.2. Consider the equation

$$\dot{x} = f(x, \varepsilon) = -x^3 + \varepsilon g(x). \tag{1.1.20}$$

If $\varepsilon = 0$, the solution with initial condition $x(0) = x_0$ is given by

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}}, \tag{1.1.21}$$

and thus $x(t)$ converges to 0 as $t \to \infty$, though much slower than in Example 1.1.1. Let us start by considering the particular case $g(x) = x$. The perturbation term being linear, it is a good idea to look for a solution of the form $x(t) = c(t)\,\mathrm{e}^{\varepsilon t}$, and we find in this way

$$x(t) = \frac{x_0\,\mathrm{e}^{\varepsilon t}}{\sqrt{1 + x_0^2 \frac{\mathrm{e}^{2\varepsilon t} - 1}{\varepsilon}}}. \tag{1.1.22}$$

If $\varepsilon \leqslant 0$, then $x(t) \to 0$ as $t \to \infty$ as before. If $\varepsilon > 0$, however, it turns out that

$$\lim_{t \to \infty} x(t) = \begin{cases} -\sqrt{\varepsilon} & \text{if } x_0 < 0 \\ 0 & \text{if } x_0 = 0 \\ \sqrt{\varepsilon} & \text{if } x_0 > 0. \end{cases} \tag{1.1.23}$$

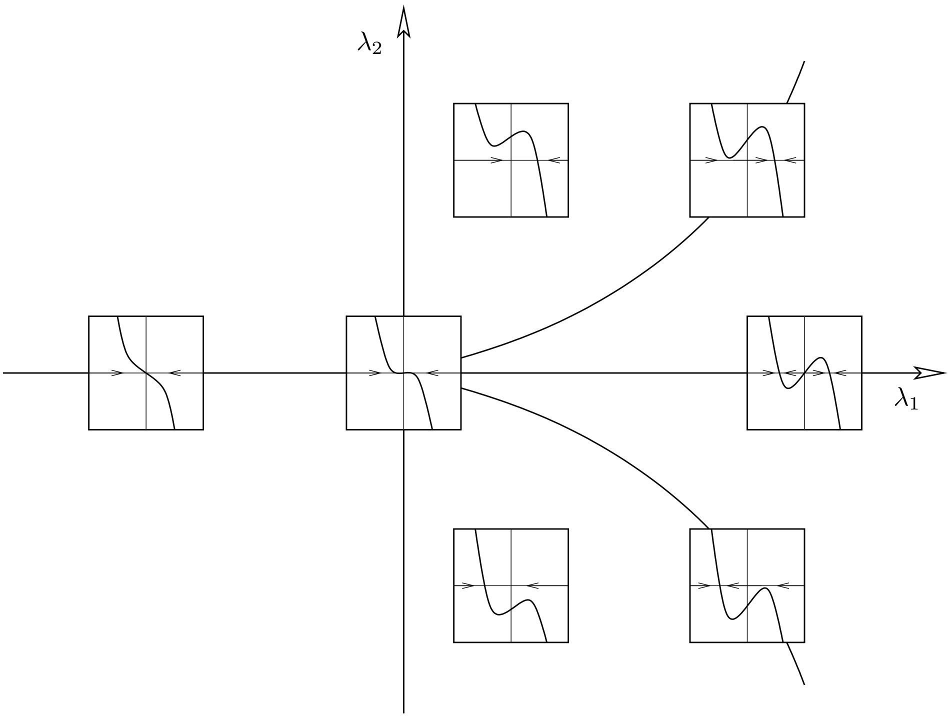

FIGURE 1.1. Graphs of the function $F_{\lambda_1,\lambda_2}(x) = -x^3 + \lambda_1 x + \lambda_2$ as a function of λ_1 and λ_2. The lines $4\lambda_1^3 = 27\lambda_2^2$, $\lambda_1 > 0$ are lines of a codimension 1 bifurcation, and the origin is a codimension 2 bifurcation point.

This behaviour differs drastically from the behaviour for $\varepsilon \leqslant 0$. Moreover, it is not obvious how the term $\sqrt{\varepsilon}$ could be obtained from an expansion of x in powers of ε. In fact, since $x(t)$ is a function of εt, any such expansion will only converge for finite values of t, and most perturbative methods introduced in Example 1.1.1 will fail in the limit $t \to \infty$.

On the other hand, the geometrical approach still works. Let us first consider the particular two-parameter family

$$\dot{x} = F_{\lambda_1,\lambda_2}(x) = -x^3 + \lambda_1 x + \lambda_2. \tag{1.1.24}$$

If $\lambda_2 = 0$, $F_{\lambda_1,0}$ vanishes only at $x = 0$ if $\lambda_1 \leqslant 0$, while $F_{\lambda_1,0}$ has three zeroes located at $-\sqrt{\lambda_1}$, 0 and $\sqrt{\lambda_1}$ if $\lambda_1 > 0$. This explains in particular the asymptotic behaviour (1.1.23). Adding the constant term λ_2 will move the graph of F vertically. If $\lambda_1 \leqslant 0$, nothing special happens, but if $\lambda_1 > 0$, two zeroes of F will disappear for sufficiently large $|\lambda_2|$ (Fig. 1.1). In the limiting case, the graph of F is tangent to the x-axis at some point $x_\star$. We thus have the conditions

$$\begin{aligned} F_{\lambda_1,\lambda_2}(x_\star) &= -x_\star^3 + \lambda_1 x_\star + \lambda_2 = 0 \\ F'_{\lambda_1,\lambda_2}(x_\star) &= -3x_\star^2 + \lambda_1 = 0. \end{aligned} \tag{1.1.25}$$

Eliminating $x_\star$ from these equations, we get the relation

$$4\lambda_1^3 = 27\lambda_2^2 \tag{1.1.26}$$

for the critical line. In summary,

- if $27\lambda_2^2 < 4\lambda_1^3$, the system has two stable equilibrium points $x^\pm$ and one unstable equilibrium point $x^0 \in (x^-, x^+)$, with

$$\lim_{t\to\infty} x(t) = \begin{cases} x^- & \text{if } x(0) < x^0 \\ x^0 & \text{if } x(0) = x^0 \\ x^+ & \text{if } x(0) > x^0; \end{cases} \tag{1.1.27}$$

- if $\lambda_1 > 0$ and $3\sqrt{3}\lambda_2 = 2\lambda_1^{3/2}$, there is a stable equilibrium x^+ and an equilibrium x^- such that

$$\lim_{t\to\infty} x(t) = \begin{cases} x^- & \text{if } x(0) \leqslant x^- \\ x^+ & \text{if } x(0) > x^-; \end{cases} \tag{1.1.28}$$

- if $\lambda_1 > 0$ and $3\sqrt{3}\lambda_2 = -2\lambda_1^{3/2}$, a similar situation occurs with the roles of x^- and x^+ interchanged;
- in all other cases, there is a unique equilibrium point attracting all solutions.

What happens for general functions g? The remarkable fact is the following: for any sufficiently smooth g and any sufficiently small ε, there exist $\lambda_1(g,\varepsilon)$ and $\lambda_2(g,\varepsilon)$ (going continuously to zero as $\varepsilon \to 0$) such that the dynamics of $\dot{x} = -x^3 + \varepsilon g(x)$ is qualitatively the same, near $x = 0$, as the dynamics of (1.1.24) with $\lambda_1 = \lambda_1(g,\varepsilon)$ and $\lambda_2 = \lambda_2(g,\varepsilon)$.

One could have expected the dynamics to depend, at least, on quadratic terms in the Taylor expansion of g. It turns out, however, that these terms can be removed by a transformation $x = y + \gamma y^2$.

The system $\dot{x} = -x^3$ is called *structurally unstable* because different perturbations, even arbitrarily small, can lead to different dynamics. It is called *singular of codimension 2* because every (smooth and small) perturbation is equivalent to a member of the 2-parameter family (1.1.24), which is called an *unfolding* (or a *versal deformation*) of the singularity.

This geometrical approach to the perturbation of vector fields is rather well established for two-dimensional systems, where all structurally stable vector fields, and all singularities of codimension 1 have been classified. We will discuss this classification in Chapter 2. Little is known on higher-dimensional systems, for which alternative methods have to be used.

1.2 Forced Oscillators

In mechanical systems, solutions are not attracted by stable equilibria, but oscillate around them. This makes the question of long-term stability an extremely difficult one, which was only solved about 50 years ago by the so-called Kolmogorov–Arnol'd–Moser (KAM) theory.

We introduce here the simplest example of this kind.

Example 1.2.1. Consider an oscillator which is being forced by a small-amplitude periodic term:

$$\ddot{x} = f(x) + \varepsilon \sin(\omega t). \tag{1.2.1}$$

We assume that $x = 0$ is a stable equilibrium of the unperturbed equation, that is, $f(0) = 0$ and $f'(0) = -\omega_0^2 < 0$.

Let us first consider the linear case, i.e. the forced harmonic oscillator:

$$\ddot{x} = -\omega_0^2 x + \varepsilon \sin(\omega t). \tag{1.2.2}$$

The theory of linear ordinary differential equations states that the general solution is obtained by superposition of a particular solution with the general solution of the homogeneous system $\ddot{x} = -\omega_0^2 x$, given by

$$x_{\mathrm{h}}(t) = x_{\mathrm{h}}(0)\cos(\omega_0 t) + \frac{\dot{x}_{\mathrm{h}}(0)}{\omega_0}\sin(\omega_0 t). \tag{1.2.3}$$

Looking for a particular solution which is periodic with the same period as the forcing term, we find

$$x_{\mathrm{p}}(t) = \frac{\varepsilon}{\omega_0^2 - \omega^2}\sin(\omega t), \tag{1.2.4}$$

and thus the general solution of (1.2.2) is given by

$$x(t) = x(0)\cos(\omega_0 t) + \frac{1}{\omega_0}\Bigl[\dot{x}(0) - \frac{\varepsilon\omega}{\omega_0^2 - \omega^2}\Bigr]\sin(\omega_0 t) + \frac{\varepsilon}{\omega_0^2 - \omega^2}\sin(\omega t). \tag{1.2.5}$$

This expression is only valid for $\omega \neq \omega_0$, but it admits a limit as $\omega \to \omega_0$, given by

$$x(t) = \Bigl[x(0) - \frac{\varepsilon t}{2\omega_0}\Bigr]\cos(\omega_0 t) + \frac{\dot{x}(0)}{\omega_0}\sin(\omega_0 t). \tag{1.2.6}$$

The solution (1.2.5) is periodic if ω/ω_0 is rational, and *quasiperiodic* otherwise. Its maximal amplitude is of the order $\varepsilon/(\omega_0^2 - \omega^2)$ when ω is close to ω_0. In the case $\omega = \omega_0$, $x(t)$ is no longer quasiperiodic, but its amplitude grows proportionally to εt. This effect, called *resonance*, is a first example showing that the long-term behaviour of an oscillator can be fundamentally changed by a small perturbation.

Before turning to the nonlinear case, we remark that equation (1.2.2) takes a particularly simple form when written in the complex variable $z = \omega_0 x + \mathrm{i}\, y$. Indeed, z satisfies the equation

$$\dot{z} = -\mathrm{i}\,\omega_0 z + \mathrm{i}\,\varepsilon\sin(\omega t), \tag{1.2.7}$$

which is easily solved by the method of variation of the constant.

Let us now consider equation (1.2.1) for more general functions f. If $\varepsilon = 0$, the system has a constant of the motion. Indeed, let $V(x)$ be a *potential function*, i.e. such that $V'(x) = -f(x)$ (for definiteness, we take $V(0) = 0$). Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigl(\frac{1}{2}\dot{x}^2 + V(x)\Bigr) = \dot{x}\ddot{x} + V'(x)\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0. \tag{1.2.8}$$

By the assumptions on f, $V(x)$ behaves like $\frac{1}{2}\omega_0^2 x^2$ near $x = 0$, and thus the level curves of $H(x,\dot{x}) = \frac{1}{2}\dot{x}^2 + V(x)$ are closed in the $(x,\dot{x})$-plane near $x = \dot{x} = 0$, which means that the origin is stable.

We can exploit this structure of phase space to introduce variables in which the equations of motion take a simpler form. For instance if $x = x(H,\psi)$, $y = y(H,\psi)$, $\psi \in [0, 2\pi)$, is a parametrization of the level curves of H, then $\dot{H} = 0$ and $\dot{\psi}$ can be expressed as a function of H and ψ. One can do even better, and introduce a parametrization by an angle

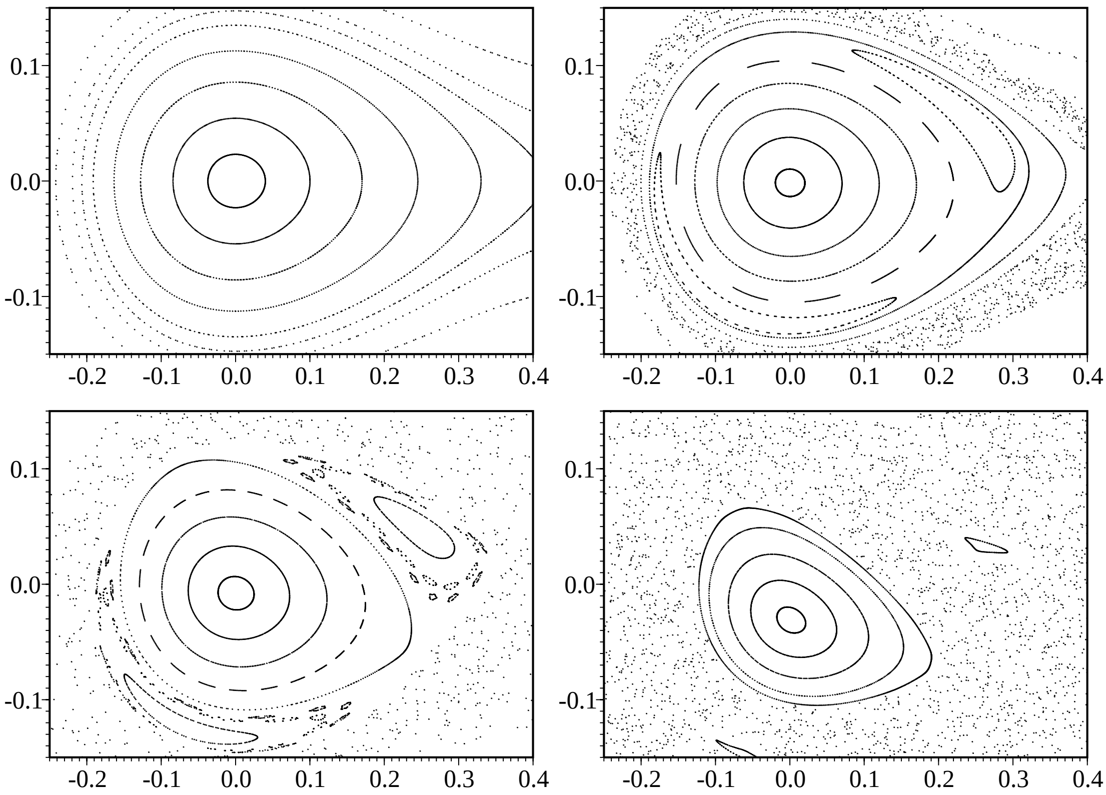

FIGURE 1.2. Poincaré maps of equation (1.2.11) in the case $f(x) = -\omega_0^2 x + x^2 - \frac{1}{2}x^3$, with $\omega_0 = 0.6$ and for increasing values of ε: from left to right and top to bottom, $\varepsilon = 0$, $\varepsilon = 0.001$, $\varepsilon = 0.005$ and $\varepsilon = 0.02$.

φ such that $\dot{\varphi}$ is constant on each level curve. In other words, there exists a coordinate transformation $(x, y) \mapsto (I, \varphi)$ such that

$$\begin{aligned} \dot{I} &= 0 \\ \dot{\varphi} &= \Omega(I). \end{aligned} \tag{1.2.9}$$

I and φ are called *action-angle variables*. $I = 0$ corresponds to $x = y = 0$ and $\Omega(0) = -\omega_0$. The variable $z = I\,\mathrm{e}^{\mathrm{i}\,\varphi}$ satisfies the equation

$$\dot{z} = \mathrm{i}\,\Omega(|z|)z \qquad \Rightarrow \qquad z(t) = \mathrm{e}^{\mathrm{i}\,\Omega(|z(0)|)}\,z(0). \tag{1.2.10}$$

Now if $\varepsilon \neq 0$, there will be an additional term in the equation, depending on z and its complex conjugate $\overline{z}$:

$$\dot{z} = \mathrm{i}\,\Omega(|z|)z + \varepsilon\sin(\omega t)g(z, \overline{z}). \tag{1.2.11}$$

If Ω were a constant, one could try to solve this equation by iterations as we did in Example 1.1.1. However, the operator $\mathcal{T}$ constructed in this way is not necessarily contracting because of the imaginary coefficient in $\dot{z}$. In fact, it turns out that the method of changes of variables works for certain special values of $\Omega(|z|)$.

Let us now find out what solutions of equation (1.2.11) look like qualitatively. Since the right-hand side is $2\pi/\omega$-periodic in t, it will be sufficient to describe the dynamics during one period. Indeed, if we define the *Poincaré map*

$$P_\varepsilon : z(0) \mapsto z(2\pi/\omega), \tag{1.2.12}$$

then z at time $n(2\pi/\omega)$ is simply obtained by applying n times the Poincaré map to $z(0)$. Let us now discuss a few properties of this map. If $\varepsilon = 0$, we know from (1.2.10) that $|z|$ is constant, and thus

$$P_0(z) = \mathrm{e}^{2\pi\,\mathrm{i}\,\Omega(|z|)/\omega}\, z. \tag{1.2.13}$$

In particular,

$$P_0(0) = 0, \qquad P_0'(0) = \mathrm{e}^{-2\pi\,\mathrm{i}\,\omega_0/\omega}\,. \tag{1.2.14}$$

For $\varepsilon > 0$, the implicit function theorem can be used to show that P_ε has a fixed point near 0, provided Ω/ω is not an integer (which would correspond to a strong resonance). One obtains that for $\Omega/\omega \notin \mathbb{Z}$, there exists a $z^\star(\varepsilon) = \mathcal{O}(\varepsilon)$ such that

$$P_\varepsilon(z^\star) = z^\star, \qquad P_\varepsilon'(z^\star) = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}, \qquad \theta = -\frac{\omega_0}{\omega} + \mathcal{O}(\varepsilon). \tag{1.2.15}$$

Thus if $\zeta = z - z^\star$, the dynamics will be described by a map of the form

$$\zeta \mapsto \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\,\zeta + G(\zeta, \overline{\zeta}, \varepsilon), \tag{1.2.16}$$

where $G(\zeta, \overline{\zeta}, \varepsilon) = \mathcal{O}(|\zeta|^2)$. In other words, the dynamics is a small perturbation of a rotation in the complex plane. Although the map (1.2.16) looks quite simple, its dynamics can be surprisingly complicated.

Fig. 1.2 shows phase portraits of the Poincaré map for a particular function f and increasing values of ε. They are obtained by plotting a large number of iterates of a few different initial conditions. For $\varepsilon = 0$, these orbits live on level curves of the constant of the motion H. As ε increases, some of these invariant curves seem to survive, so that the point in their center (which is $z^\star$) is surrounded by an "elliptic island". This means that stable motions still exist. However, more and more of the invariant curves are destroyed, so that the island is gradually invaded by chaotic orbits. A closer look at the island shows that it is not foliated by invariant curves, but contains more complicated structures, also called resonances.

One of the aims of Chapter 3 will be to develop methods to study perturbed systems similar to (1.2.1), and to find conditions under which stable equilibria of the unperturbed system remain stable when a small perturbation is added.

1.3 Singular Perturbations: The Van der Pol Oscillator

Singular perturbation theory considers systems of the form $\dot{x} = f(x, \varepsilon)$ in which f behaves singularly in the limit $\varepsilon \to 0$. A simple example of such a system is the Van der Pol oscillator in the large damping limit.

Example 1.3.1. The Van der Pol oscillator is a second order system with nonlinear damping, of the form

$$\ddot{x} + \alpha(x^2 - 1)\dot{x} + x = 0. \tag{1.3.1}$$

The special form of the damping (which can be realized by an electric circuit) has the effect of decreasing the amplitude of large oscillations, while increasing the amplitude of small oscillations.

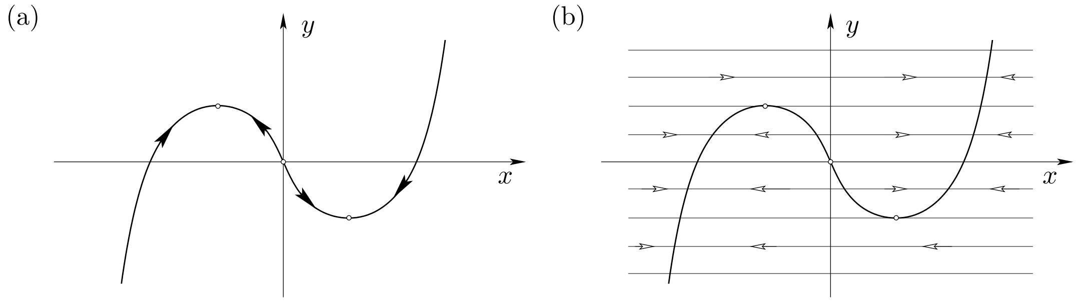

FIGURE 1.3. Behaviour of the Van der Pol equation in the singular limit $\varepsilon \to 0$, (a) on the slow time scale $t' = \sqrt{\varepsilon}t$, given by (1.3.4), and (b) on the fast time scale $t'' = t/\sqrt{\varepsilon}$, see (1.3.6).

We are interested in the behaviour for large α. There are several ways to write (1.3.1) as a first order system. For our purpose, a convenient representation is

$$\begin{aligned} \dot{x} &= \alpha\Big(y + x - \frac{x^3}{3}\Big) \\ \dot{y} &= -\frac{x}{\alpha}. \end{aligned} \tag{1.3.2}$$

One easily checks that this system is equivalent to (1.3.1) by computing $\ddot{x}$. If α is very large, x will move quickly, while y changes slowly. To analyse the limit $\alpha \to \infty$, we introduce a small parameter $\varepsilon = 1/\alpha^2$ and a "slow time" $t' = t/\alpha = \sqrt{\varepsilon}t$. Then (1.3.2) can be rewritten as

$$\begin{aligned} \varepsilon\frac{\mathrm{d}x}{\mathrm{d}t'} &= y + x - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -x. \end{aligned} \tag{1.3.3}$$

In the limit $\varepsilon \to 0$, we obtain the system

$$\begin{aligned} 0 &= y + x - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t'} &= -x, \end{aligned} \tag{1.3.4}$$

which is no longer a system of differential equations. In fact, the solutions are constrained to move on the curve $\mathcal{C} : y = \frac{1}{3}x^3 - x$, and eliminating y from the system we have

$$-x = \frac{\mathrm{d}y}{\mathrm{d}t'} = (x^2 - 1)\frac{\mathrm{d}x}{\mathrm{d}t'} \qquad \Rightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t'} = -\frac{x}{x^2 - 1}. \tag{1.3.5}$$

The dynamics is shown in Fig. 1.3a. Another possibility is to introduce the "fast time" $t'' = \alpha t = t/\sqrt{\varepsilon}$. Then (1.3.2) becomes

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t''} &= y + x - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t''} &= -\varepsilon x. \end{aligned} \tag{1.3.6}$$

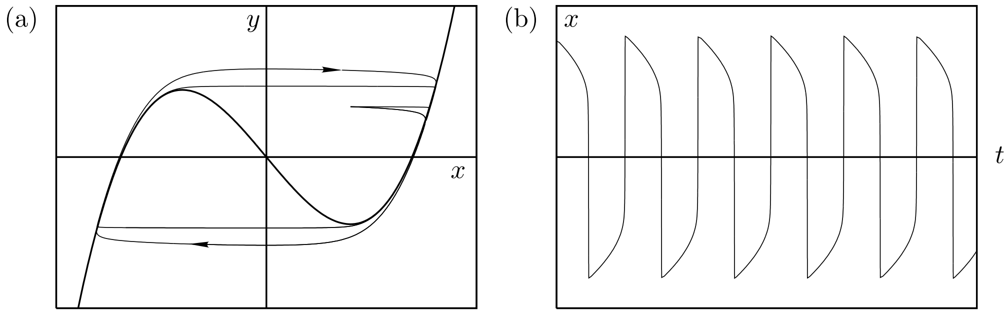

FIGURE 1.4. (a) Two solutions of the Van der Pol equations (1.3.2) (light curves) for the same initial condition $(1, 0.5)$, for $\alpha = 5$ and $\alpha = 20$. The heavy curve is the curve $\mathcal{C} : y = \frac{1}{3}x^3 - x$. (b) The graph of $x(t)$ ($\alpha = 20$) displays relaxation oscillations.

In the limit $\varepsilon \to 0$, we get the system

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t''} &= y + x - \frac{x^3}{3} \\ \frac{\mathrm{d}y}{\mathrm{d}t''} &= 0. \end{aligned} \tag{1.3.7}$$

In this case, y is a constant and acts as a parameter in the equation for x. Some orbits are shown in Fig. 1.3b.

Of course, the systems (1.3.2), (1.3.3) and (1.3.6) are strictly equivalent for $\varepsilon > 0$. They only differ in the singular limit $\varepsilon \to 0$. The dynamics for small but positive ε can be understood by sketching the vector field. Let us note that

- $\dot{x}$ is positive if (x, y) lies above the curve $\mathcal{C}$ and negative when it lies below; this curve separates the plane into regions where x moves to the right or to the left, and the orbit must cross the curve vertically;
- $\mathrm{d}y/\mathrm{d}x$ is very small unless the orbit is close to the curve $\mathcal{C}$, so that the orbits will be almost horizontal except near this curve;
- orbits move upward if $x < 0$ and downward if $x > 0$.

The resulting orbits are shown in Fig. 1.4a. An orbit starting somewhere in the plane will first approach the curve $\mathcal{C}$ on a nearly horizontal path, in a time t of order $1/\alpha$. Then it will track the curve at a small distance until it reaches a turning point, after a time t of order α. Since the equations forbid it to follow $\mathcal{C}$ beyond this point, the orbit will jump to another branch of $\mathcal{C}$, where the behaviour repeats. The graph of $x(t)$ thus contains some parts with a small slope and others with a large slope (Fig. 1.4b). This phenomenon is called a *relaxation oscillation*.

The system (1.3.3) is called a *slow-fast* system, and is a particular case of a singularly perturbed equation. We will discuss some properties of these systems in Chapter 4.

Chapter 2

Bifurcations and Unfolding

The qualitative theory of two-dimensional vector fields is relatively well developed, since all structurally stable systems and all singularities of codimension 1 have been identified. Thus we know which systems are well-behaved under small perturbations, and most of the other systems are qualitatively well understood.

In this chapter we will consider ordinary differential equations of the form

$$\dot{x} = f(x), \qquad f \in \mathcal{C}^k(\mathcal{D}, \mathbb{R}^2), \tag{2.0.1}$$

where $\mathcal{D}$ is an open domain in $\mathbb{R}^2$ and the differentiability k is at least 1 (we will often need a higher differentiability). f is called a *vector field*. Equivalently, equation (2.0.1) can be written in components as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2). \end{aligned} \tag{2.0.2}$$

In fact, one may also consider the case where $\mathcal{D}$ is a compact, two-dimensional manifold, such as the 2-sphere $\mathbb{S}^2$ or the 2-torus $\mathbb{T}^2$. This actually simplifies some of the problems, but the main results are essentially the same.

We will start by establishing some properties of invariant sets of the ODE (2.0.1), before proceeding to the classification of structurally stable and unstable vector fields.

2.1 Invariant Sets of Planar Flows

Results from basic analysis show that the condition $f \in \mathcal{C}^1(\mathcal{D}, \mathbb{R}^2)$ is sufficient to guarantee the existence of a unique solution $x(t)$ of (2.0.1) for every initial condition $x(0) \in \mathcal{D}$. This solution can be extended to a maximal open t-interval $I \ni 0$; I may not be equal to $\mathbb{R}$, but in this case $x(t)$ must tend to the boundary of $\mathcal{D}$ as t approaches the boundary of I.

Definition 2.1.1. *Let $x_0 \in \mathcal{D}$ and let $x : I \to \mathcal{D}$ be the unique solution of* (2.0.1) *with initial condition $x(0) = x_0$ and maximal interval of existence I.*

- *the* orbit *through x_0 is the set $\{x \in \mathcal{D} \colon x = x(t), t \in I\}$;*
- *the* flow *of* (2.0.1) *is the map*

$$\varphi : (x_0, t) \mapsto \varphi_t(x_0) = x(t). \tag{2.1.1}$$

The uniqueness property implies that

$$\varphi_0(x_0) = x_0 \qquad \text{and} \qquad \varphi_t(\varphi_s(x_0)) = \varphi_{t+s}(x_0) \tag{2.1.2}$$

for all $x_0 \in \mathcal{D}$ and all $t, s \in \mathbb{R}$ for which these quantities are defined.

Definition 2.1.2. *A set $\mathcal{S} \in \mathcal{D}$ is called*

- positively invariant *if $\varphi_t(\mathcal{S}) \subset \mathcal{S}$ for all $t \geqslant 0$;*
- negatively invariant *if $\varphi_t(\mathcal{S}) \subset \mathcal{S}$ for all $t \leqslant 0$;*
- invariant *if $\varphi_t(\mathcal{S}) = \mathcal{S}$ for all $t \in \mathbb{R}$.*

Note that if $\mathcal{S}$ is positively invariant, then $\varphi_t(x_0)$ exists for all $t \in [0, \infty)$ whenever $x_0 \in \mathcal{S}$, and similar properties hold in the other cases. A sufficient condition for $\mathcal{S}$ to be positively invariant is that the vector field f be directed inward $\mathcal{S}$ on the boundary $\partial\mathcal{S}$.

We shall now introduce some particularly important invariant sets.

Definition 2.1.3. *Assume that $\varphi_t(x)$ exists for all $t \geqslant 0$. The* ω-limit set *of x for φ_t, denoted $\omega(x)$, is the set of points $y \in \mathcal{D}$ such that there exists a sequence $\{t_n\}_{n\geqslant 0}$ such that $t_n \to \infty$ and $\varphi_{t_n}(x) \to y$ as $n \to \infty$. The* α-limit set *$\alpha(x)$ is defined in a similar way, with $t_n \to -\infty$.*

The terminology is due to the fact that α and ω are the first and last letters of the Greek alphabet.

Proposition 2.1.4. *Let $\mathcal{S}$ be positively invariant and compact. Then for any $x \in \mathcal{S}$,*

1. *$\omega(x) \neq \emptyset$;*
2. *$\omega(x)$ is closed;*
3. *$\omega(x)$ is invariant under φ_t;*
4. *$\omega(x)$ is connected.*

PROOF:

1. Choose a sequence $t_n \to \infty$ and set $x_n = \varphi_{t_n}(x)$. Since $\mathcal{S}$ is compact, $\{x_n\}$ has a convergent subsequence, and its limit is a point in $\omega(x)$.
2. Pick any $y \notin \omega(x)$. There must exist a neighbourhood $\mathcal{U}$ of y and $T \geqslant 0$ such that $\varphi_t(x) \notin \mathcal{U}$ for all $t \geqslant T$. This shows that the complement of $\omega(x)$ is open, and thus $\omega(x)$ is closed.
3. Let $y \in \omega(x)$ and let $t_n \to \infty$ be an increasing sequence such that $\varphi_{t_n}(x) \to y$ as $n \to \infty$. For any $s \in \mathbb{R}$, (2.1.2) implies that $\varphi_s(\varphi_{t_n}(x)) = \varphi_{s+t_n}(x)$ exists for all $s \geqslant -t_n$. Taking the limit $n \to \infty$, we obtain by continuity that $\varphi_s(y)$ exists for all $s \in \mathbb{R}$. Moreover, since $\varphi_{s+t_n}(x)$ converges to $\varphi_s(y)$ as $t_n \to \infty$, we conclude that $\varphi_s(y) \in \omega(x)$, and therefore $\omega(x)$ is invariant.
4. Suppose by contradiction that $\omega(x)$ is not connected. Then there exist two open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\overline{\mathcal{U}}_1 \cap \overline{\mathcal{U}}_2 = \emptyset$, $\omega(x) \subset \mathcal{U}_1 \cup \mathcal{U}_2$, and $\omega(x) \cap \mathcal{U}_i \neq \emptyset$ for $i = 1, 2$. By continuity of $\varphi_t(x)$, given $T > 0$, there exists $t > T$ such that $\varphi_t(x) \in \mathcal{K} := \mathcal{S} \setminus (\mathcal{U}_1 \cup \mathcal{U}_2)$. We can thus find a sequence $t_n \to \infty$ such that $\varphi_{t_n}(x) \in \mathcal{K}$, which must admit a subsequence converging to some $y \in \mathcal{K}$. But this would imply $y \in \omega(x)$, a contradiction. $\square$

Typical examples of ω-limit sets are attracting equilibrium points and periodic orbits. We will start by examining these in more detail before giving the general classification of ω-limit sets.

2.1.1 Equilibrium Points

Definition 2.1.5. *$x^\star \in \mathcal{D}$ is called an* equilibrium point *(or a* singular point*) of the system $\dot{x} = f(x)$ if $f(x^\star) = 0$. Equivalently, we have $\varphi_t(x^\star) = x^\star$ for all $t \in \mathbb{R}$, and hence $\{x^\star\}$ is invariant.*

We would like to characterize the dynamics near $x^\star$. Assuming $f \in \mathcal{C}^2$, we can Taylor-expand f to second order around $x^\star$: if $y = x - x^\star$, then

$$\dot{y} = f(x^\star + y) = Ay + g(y), \tag{2.1.3}$$

where A is the *Jacobian matrix* of f at $x^\star$, defined by

$$A = \frac{\partial f}{\partial x}(x^\star) := \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1}(x^\star) & \dfrac{\partial f_1}{\partial x_2}(x^\star) \\ \dfrac{\partial f_2}{\partial x_1}(x^\star) & \dfrac{\partial f_2}{\partial x_2}(x^\star) \end{pmatrix} \tag{2.1.4}$$

and $g(y) = \mathcal{O}(\|y\|^2)$ as $y \to 0$. The matrix A has eigenvalues a_1, a_2, which are either both real, or complex conjugate.

Definition 2.1.6. *The equilibrium point $x^\star$ is called*
- hyperbolic *if* $\operatorname{Re} a_1 \neq 0$ *and* $\operatorname{Re} a_2 \neq 0$*;*
- non-hyperbolic *if* $\operatorname{Re} a_1 = 0$ *or* $\operatorname{Re} a_2 = 0$*;*
- elliptic *if* $\operatorname{Re} a_1 = \operatorname{Re} a_2 = 0$ *(but one usually requires* $\operatorname{Im} a_1 \neq 0$ *and* $\operatorname{Im} a_2 \neq 0$*).*

In order to understand the meaning of these definitions, let us start by considering the *linearization* of (2.1.3):

$$\dot{y} = Ay \qquad \Rightarrow \qquad y(t) = \mathrm{e}^{At}\, y(0), \tag{2.1.5}$$

where the *exponential of At* is defined by the absolutely convergent series

$$\mathrm{e}^{At} := \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k. \tag{2.1.6}$$

The simplest way to compute e^{At} is to choose a basis in which A is in Jordan canonical form. There are four qualitatively different (real) canonical forms:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \qquad \begin{pmatrix} a & -\omega \\ \omega & a \end{pmatrix} \qquad \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \qquad \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \tag{2.1.7}$$

where we assume that $a_1 \neq a_2$ and $\omega \neq 0$. They correspond to the following situations:

1. If a_1 and a_2 are both real and different, then

$$\mathrm{e}^{At} = \begin{pmatrix} \mathrm{e}^{a_1 t} & 0 \\ 0 & \mathrm{e}^{a_2 t} \end{pmatrix} \qquad \Rightarrow \qquad \begin{aligned} y_1(t) &= \mathrm{e}^{a_1 t}\, y_1(0) \\ y_2(t) &= \mathrm{e}^{a_2 t}\, y_2(0). \end{aligned} \tag{2.1.8}$$

The orbits are curves of the form $y_2 = c y_1^{a_2/a_1}$, and $x^\star$ is called a *node* if a_1 and a_2 have the same sign, and a *saddle* if they have opposite signs (Fig. 2.1a and b).

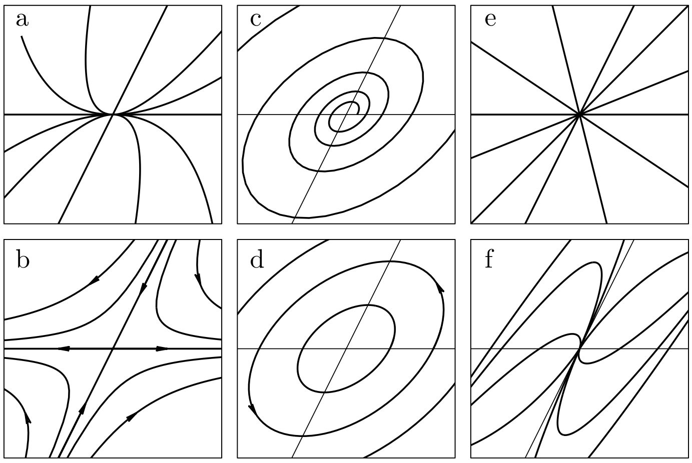

FIGURE 2.1. Phase portraits of a linear two–dimensional system: (a) node, (b) saddle, (c) focus, (d) center, (e) degenerate node, (f) improper node.

2. If $a_1 = \overline{a_2} = a + \mathrm{i}\,\omega$ with $\omega \neq 0$, then

$$\mathrm{e}^{At} = \mathrm{e}^{at}\begin{pmatrix}\cos\omega t & -\sin\omega t\\ \sin\omega t & \cos\omega t\end{pmatrix} \quad\Rightarrow\quad \begin{aligned} y_1(t) &= \mathrm{e}^{at}(y_1(0)\cos\omega t - y_2(0)\sin\omega t)\\ y_2(t) &= \mathrm{e}^{at}(y_1(0)\sin\omega t + y_2(0)\cos\omega t).\end{aligned} \tag{2.1.9}$$

The orbits are spirals or ellipses, and $x^\star$ is called a *focus* is $a \neq 0$ and a *center* if $a = 0$ (Fig. 2.1c and d).

3. If $a_1 = a_2 = a$ and A admits two independent eigenvectors, then

$$\mathrm{e}^{At} = \mathrm{e}^{at}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \quad\Rightarrow\quad \begin{aligned} y_1(t) &= \mathrm{e}^{at}\,y_1(0)\\ y_2(t) &= \mathrm{e}^{at}\,y_2(0).\end{aligned} \tag{2.1.10}$$

The orbits are straight lines, and $x^\star$ is called a *degenerate node* (Fig. 2.1e).

4. If $a_1 = a_2 = a$ but A admits only one independent eigenvector, then

$$\mathrm{e}^{At} = \mathrm{e}^{at}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix} \quad\Rightarrow\quad \begin{aligned} y_1(t) &= \mathrm{e}^{at}(y_1(0) + y_2(0)t)\\ y_2(t) &= \mathrm{e}^{at}\,y_2(0).\end{aligned} \tag{2.1.11}$$

$x^\star$ is called an *improper node* (Fig. 2.1f).

The above discussion shows that the asymptotic behaviour depends only on the sign of the real parts of the eigenvalues of A. This property carries over to the nonlinear equation (2.1.3) if $x^\star$ is hyperbolic, while nonhyperbolic equilibria are more sensitive to nonlinear perturbations. We will demonstrate this by considering first the case of nodes/foci, and then the case of saddles.

Proposition 2.1.7. *Assume that* $\operatorname{Re} a_1 < 0$ *and* $\operatorname{Re} a_2 < 0$*. Then there exists a neighbourhood* $\mathcal{U}$ *of* $x^\star$ *such that* $\omega(x) = x^\star$ *for all* $x \in \mathcal{U}$*.*

PROOF: The proof is a particular case of a theorem due to Liapunov. Consider the case where a_1 and a_2 are real and different. Then system (2.1.3) can be written in appropriate coordinates as

$$\begin{aligned}\dot y_1 &= a_1 y_1 + g_1(y)\\ \dot y_2 &= a_2 y_2 + g_2(y),\end{aligned}$$

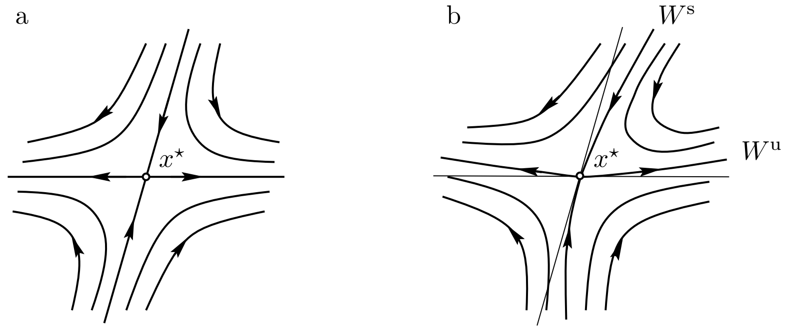

FIGURE 2.2. Orbits near a hyperbolic fixed point: (a) orbits of the linearized system, (b) orbits of the nonlinear system with local stable and unstable manifolds.

where $|g_j(y)| \leqslant M\|y\|^2$, $j = 1, 2$, for sufficiently small $\|y\|$ and for some constant $M > 0$. Given a solution $y(t)$ with initial condition $y(0) = y_0$, we define the (Liapunov) function

$$V(t) = \|y(t)\|^2 = y_1(t)^2 + y_2(t)^2.$$

Differentiating with respect to time, we obtain

$$\begin{aligned} \dot V(t) &= 2\big[y_1(t)\dot y_1(t) + y_2(t)\dot y_2(t)\big] \\ &= 2\big[a_1 y_1(t)^2 + a_2 y_2(t)^2 + y_1(t)g_1(y(t)) + y_2(t)g_2(y(t))\big]. \end{aligned}$$

If, for instance, $a_1 > a_2$, we conclude from the properties of the g_j that

$$\dot V(t) \leqslant \big[a_1 + M(|y_1(t)| + |y_2(t)|)\big]V(t).$$

Since $a_1 < 0$, there exists a constant $r > 0$ such that term in brackets is strictly negative for $0 \leqslant \|y(t)\| \leqslant r$. This shows that the disc of radius r is positively invariant. Any solution starting in this disc must converge to $y = 0$, since otherwise we would contradict the fact that $\dot V < 0$ if $y \neq 0$. The other cases can be proved in a similar way. □

Corollary 2.1.8. *Assume that* $\operatorname{Re} a_1 > 0$ *and* $\operatorname{Re} a_2 > 0$. *Then there exists a neighbourhood* $\mathcal{U}$ *of* $x^\star$ *such that* $\alpha(x) = x^\star$ *for all* $x \in \mathcal{U}$.

PROOF: By changing t into $-t$, we transform the situation into the previous one. □

The remaining hyperbolic case is the saddle point. In the linear case, we saw that the eigenvectors define two invariant lines, on which the motion is either contracting or expanding. This property can be generalized to the nonlinear case.

Theorem 2.1.9 (Stable Manifold Theorem). *Assume that* $a_1 < 0 < a_2$. *Then*

- *there exists a curve* $W^{\mathrm{s}}(x^\star)$*, tangent in* $x^\star$ *to the eigenspace of* a_1*, such that* $\omega(x) = x^\star$ *for all* $x \in W^{\mathrm{s}}(x^\star)$*;* W^{s} *is called the* stable manifold *of* $x^\star$*;*
- *there exists a curve* $W^{\mathrm{u}}(x^\star)$*, tangent in* $x^\star$ *to the eigenspace of* a_2*, such that* $\alpha(x) = x^\star$ *for all* $x \in W^{\mathrm{u}}(x^\star)$*;* W^{u} *is called the* unstable manifold *of* $x^\star$.

The situation is sketched in Fig. 2.2. We conclude that there are three qualitatively different types of hyperbolic points: *sinks*, which attract all orbits from a neighbourhood, *sources*, which repel all orbits from a neighbourhood, and *saddles*, which attract orbits from one direction, and repel them into another one. These three situations are robust to small perturbations.

2.1.2 Periodic Orbits

Definition 2.1.10. *A* periodic orbit *is an orbit forming a closed curve Γ in $\mathcal{D}$. Equivalently, if $x_0 \in \mathcal{D}$ is not an equilibrium point, and $\varphi_T(x_0) = x_0$ for some $T > 0$, then the orbit of x_0 is a periodic orbit with period T. T is called the* least period *of Γ is $\varphi_t(x_0) \neq x_0$ for $0 < t < T$.*

We denote by $\gamma(t) = \varphi_t(x_0)$ a periodic solution living on Γ. In order to analyse the dynamics near Γ, we introduce the variable $y = x - \gamma(t)$ which satisfies the equation

$$\dot{y} = f(\gamma(t) + y) - f(\gamma(t)). \tag{2.1.12}$$

We start again by considering the linearization of this equation around $y = 0$, given by

$$\dot{y} = A(t)y \qquad \text{where } A(t) = \frac{\partial f}{\partial x}(\gamma(t)). \tag{2.1.13}$$

The solution can be written in the form $y(t) = U(t)y(0)$, where the *principal solution* $U(t)$ is a matrix-valued function solving the linear equation

$$\dot{U} = A(t)U, \qquad U(0) = \mathbb{1}. \tag{2.1.14}$$

Such an equation is difficult to solve in general. In the present case, however, there are several simplifications. First of all, since $A(t) = A(t + T)$ for all t, Floquet's theorem allows us to write

$$U(t) = P(t)\,\mathrm{e}^{Bt}, \tag{2.1.15}$$

where B is a constant matrix, and $P(t)$ is a T-periodic function of time, with $P(0) = \mathbb{1}$. The asymptotic behaviour of $y(t)$ thus depends only on the eigenvalues of B, which are called *characteristic exponents* of Γ. $U(T) = \mathrm{e}^{BT}$ is called the *monodromy matrix*, and its eigenvalues are called the *characteristic multipliers*.

Proposition 2.1.11. *The characteristic exponents of Γ are given by*

$$0 \qquad \textit{and} \qquad \frac{1}{T}\int_0^T \Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big)(\gamma(t))\,\mathrm{d}t. \tag{2.1.16}$$

PROOF: We first observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\dot{\gamma}(t) = \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) = \frac{\partial f}{\partial x}(\gamma(t))\dot{\gamma}(t) = A(t)\dot{\gamma}(t).$$

Thus $\dot{\gamma}(t)$ is a particular solution of (2.1.13) and can be written, by (2.1.15), as $\dot{\gamma}(t) = P(t)\,\mathrm{e}^{Bt}\,\dot{\gamma}(0)$. It follows by periodicity of γ that

$$\dot{\gamma}(0) = \dot{\gamma}(T) = P(T)\,\mathrm{e}^{BT}\,\dot{\gamma}(0) = \mathrm{e}^{BT}\,\dot{\gamma}(0).$$

In other words, $\dot{\gamma}(0)$ is an eigenvector of e^{BT} with eigenvalue 1, which implies that B has an eigenvalue equal to 0.

Next we use the fact that the principal solution satisfies the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}\det U(t) = \operatorname{Tr} A(t)\det U(t),$$

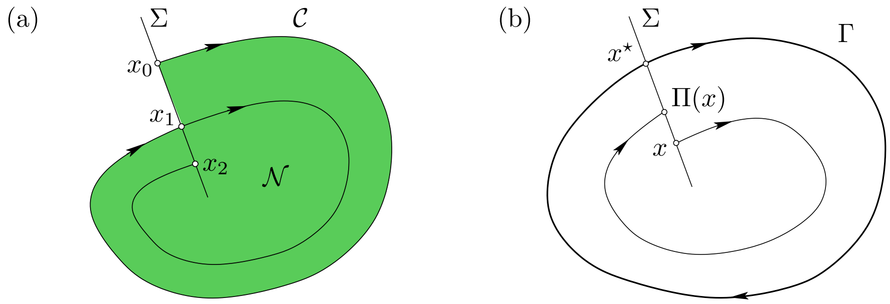

FIGURE 2.3. (a) Proof of Lemma 2.1.13: The orbit of x_0 defines a positively invariant set and crosses the transverse arc Σ in a monotone sequence of points. (b) Definition of a Poincaré map Π associated with the periodic orbit Γ.

as a consequence of the fact that $\det(\mathbb{1} + \varepsilon B) = 1 + \varepsilon \operatorname{Tr} B + \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$. Since $\det U(0) = 1$, we obtain

$$\det \mathrm{e}^{BT} = \det U(T) = \exp\Bigl\{\int_0^T \operatorname{Tr} A(t)\,\mathrm{d}t\Bigr\}.$$

But $\det \mathrm{e}^{BT}$ is the product of the characteristic multipliers, and thus $\log\det \mathrm{e}^{BT}$ is T times the sum of the characteristic exponents, one of which we already know is equal to 0. The result follows from the definition of A. □

In the linear case, the second characteristic exponent in (2.1.16) determines the stability: if it is negative, then Γ attracts other orbits, and if it is positive, then Γ repels other orbits. The easiest way to extend these properties to the nonlinear case is geometrical. It relies heavily on a "geometrically obvious" property of planar curves, which is notoriously hard to prove.

Theorem 2.1.12 (Jordan Curve Theorem). *A closed curve in $\mathbb{R}^2$ which does not intersect itself separates $\mathbb{R}^2$ into two connected components, one bounded, called the* interior *of the curve, the other unbounded, called the* exterior *of the curve.*

Let Σ be an arc (a piece of a differentiable curve) in the plane. We say that Σ is *transverse* to the vector field f if at any point of Σ, the tangent vector to Σ and f are not collinear.

Lemma 2.1.13. *Assume the vector field is transverse to Σ. Let $x_0 \in \Sigma$ and denote by $x_1, x_2, \dots$ the successive intersections, as time increases, of the orbit of x_0 with Σ (as long as they exist). Then for every i, x_i lies between x_{i-1} and x_{i+1} on Σ.*

PROOF: Assume the orbit of x_0 intersects Σ for the first (positive) time in x_1 (otherwise there is nothing to prove). Consider the curve $\mathcal{C}$ consisting of the orbit between x_0 and x_1, and the piece of Σ between the same points (Fig. 2.3a). Then $\mathcal{C}$ admits an interior $\mathcal{N}$, which is either positively invariant or negatively invariant. In the first case, we conclude that x_2, if it exists, lies inside $\mathcal{N}$. In the second case, x_2 must lie outside $\mathcal{N}$ because otherwise the negative orbit of x_2 would have to leave $\mathcal{N}$. □

Assume now that Σ is a small segment, intersecting the periodic orbit Γ transversally at exactly one point $x^\star$. For each $x \in \Sigma$, we define a *first return time*

$$\tau(x) = \inf\bigl\{t > 0 \colon \varphi_t(x) \in \Sigma\bigr\} \in (0, \infty]. \tag{2.1.17}$$

By continuity of the flow, $\tau(x) < \infty$ for x sufficiently close to $x^\star$ (Fig. 2.3b). The *Poincaré map* of Γ associated with Σ is the map

$$\begin{aligned} \Pi : \quad \{x \in \Sigma \colon \tau(x) < \infty\} &\to \Sigma \\ x &\mapsto \varphi_{\tau(x)}(x). \end{aligned} \tag{2.1.18}$$

Theorem 2.1.14. *The Poincaré map has the following properties:*

- Π *is a monotone increasing* $\mathcal{C}^1$ *map in a neighbourhood of* $x^\star$*;*
- *the multipliers of* Γ *are* 1 *and* $\Pi'(x^\star)$*;*
- *if* $\Pi'(x^\star) < 1$*, then there exists a neighbourhood* $\mathcal{U}$ *of* Γ *such that* $\omega(x) = \Gamma$ *for every* $x \in \mathcal{U}$*;*
- *if* $\Pi'(x^\star) > 1$*, then there exists a neighbourhood* $\mathcal{U}$ *of* Γ *such that* $\alpha(x) = \Gamma$ *for every* $x \in \mathcal{U}$*.*

PROOF:

- The differentiability of Π is a consequence of the implicit function theorem. Define some ordering on Σ, and consider two points $x < y$ on Σ. We want to show that $\Pi(x) < \Pi(y)$. As in the proof of Lemma 2.1.13, the orbit of each point defines a positively or negatively invariant set $\mathcal{N}$, resp. $\mathcal{M}$. Assume for instance that $\mathcal{N}$ is positively invariant and $y \in \mathcal{N}$. Then $\Pi(y) \in \mathcal{N}$ and the conclusion follows. The other cases are similar.
- We already saw in Proposition 2.1.11 that $\mathrm{e}^{BT} f(x^\star) = f(x^\star)$. Let v be a tangent vector to Σ of unit length, a a sufficiently small scalar, and consider the relation

  $$\Pi(x^\star + av) = \varphi_{\tau(x^\star+av)}(x^\star + av).$$

  Differentiating this with respect to a and evaluating at $a = 0$, we get

  $$\Pi'(x^\star)v = \mathrm{e}^{BT} v + \tau'(x^\star) f(x^\star).$$

  This shows that in the basis $(f(x^\star), v)$, the monodromy matrix has the representation

  $$\mathrm{e}^{BT} = \begin{pmatrix} 1 & -\tau'(x^\star) \\ 0 & \Pi'(x^\star) \end{pmatrix}.$$

- Assume that $\Pi'(x^\star) < 1$. Then Π is a contraction in some neighbourhood of $x^\star$. This implies that there exists an open interval (x_1, x_2) containing $x^\star$ which is mapped strictly into itself. Thus the orbits of x_1 and x_2, between two successive intersections with Σ, define a positively invariant set $\mathcal{N}$. We claim that $\omega(x) = \Gamma$ for any $x \in \mathcal{N}$. Let Σ_0 denote the piece of Σ between x_1 and x_2. If $x \in \Sigma_0$, then the iterates $\Pi^n(x)$ converge to $x^\star$ as $n \to \infty$, which shows that $x^\star \in \omega(x)$. It also shows that $x^\star$ is the only point of Σ which belongs to $\omega(x)$. If $x \notin \Sigma_0$, there exists by continuity of the flow a time $t \in (0, T)$ such that $x \in \varphi_t(\Sigma_0)$. Hence the sequence $\{\Pi^n \circ \varphi_{T-t}(x)\}$ converges to $x^\star$. Using a similar combination of φ_t and Π^n, one can construct a sequence converging to any $y \in \Gamma$.
- The case $\Pi'(x^\star) > 1$ can be treated in the same way by considering Π^{-1}. □

Definition 2.1.15. *The periodic orbit* Γ *is* hyperbolic *if* $\Pi'(x^\star) \neq 1$*. In other words,* Γ *is hyperbolic if*

$$\int_0^T \Big(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Big)(\gamma(t))\,\mathrm{d}t \neq 0. \tag{2.1.19}$$

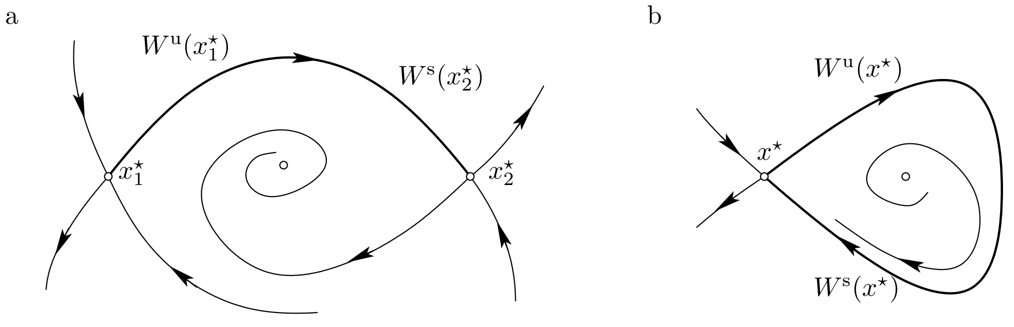

FIGURE 2.4. Examples of saddle connections: (a) heteroclinic connection, (b) homoclinic connection.

2.1.3 The Poincaré–Bendixson Theorem

Up to now, we have encountered equilibrium points and periodic orbits as possible limit sets. There exists a third kind of orbit that may be part of a limit set.

Definition 2.1.16. *Let $x_1^\star$ and $x_2^\star$ be two saddles. Assume that their stable and unstable manifolds intersect, i.e. there exists a point $x_0 \in W^u(x_1^\star) \cap W^s(x_2^\star)$. Then the orbit Γ of x_0 must be contained in $W^u(x_1^\star) \cap W^s(x_2^\star)$ and is called a* saddle connection*. This implies that $\alpha(x) = x_1^\star$ and $\omega(x) = x_2^\star$ for any $x \in \Gamma$. The connection is called* heteroclinic *if $x_1^\star \neq x_2^\star$ and* homoclinic *if $x_1^\star = x_2^\star$.*

Examples of saddle connections are shown in Fig. 2.4. The remarkable fact is that this is the exhaustive list of all possible limit sets in two dimensions, as shown by the following theorem.

Theorem 2.1.17 (Poincaré–Bendixson). *Let $\mathcal{S}$ be a compact positively invariant region containing a finite number of equilibrium points. For any $x \in \mathcal{S}$, one of the following possibilities holds:*

1. *$\omega(x)$ is an equilibrium point;*
2. *$\omega(x)$ is a periodic orbit;*
3. *$\omega(x)$ consists of a finite number of equilibrium points $x_1^\star, \dots, x_n^\star$ and orbits Γ_k with $\alpha(\Gamma_k) = x_i^\star$ and $\omega(\Gamma_k) = x_j^\star$.*

PROOF: The proof is based on the following observations:

1. **Let $\Sigma \subset \mathcal{S}$ be an arc transverse to the vector field. Then $\omega(x)$ intersects Σ in one point at most.**
   Assume by contradiction that $\omega(x)$ intersects Σ in two points y, z. Then there exist sequences of points $\{y_n\} \in \Sigma$ and $\{z_n\} \in \Sigma$ in the orbit of x such that $y_n \to y$ and $z_n \to z$ as $n \to \infty$. However, this would contradict Lemma 2.1.13.
2. **If $\omega(x)$ does not contain equilibrium points, then it is a periodic orbit.**
   Choose $y \in \omega(x)$ and $z \in \omega(y)$. The definition of ω-limit sets implies that $z \in \omega(x)$. Since $\omega(x)$ is closed, invariant, and contains no equilibria, $z \in \omega(x)$ is not an equilibrium. We can thus construct a transverse arc Σ through z. The orbit of y intersects Σ in a monotone sequence $\{y_n\}$ with $y_n \to z$ as $n \to \infty$. Since $y_n \in \omega(x)$, we must have $y_n = z$ for all n by point 1. Hence the orbit of y must be a closed curve. Now take a transverse arc Σ' through y. By point 1., $\omega(x)$ intersects Σ' only at y. Since $\omega(x)$ is connected, invariant, and without equilibria, it must be identical with the periodic orbit through y.

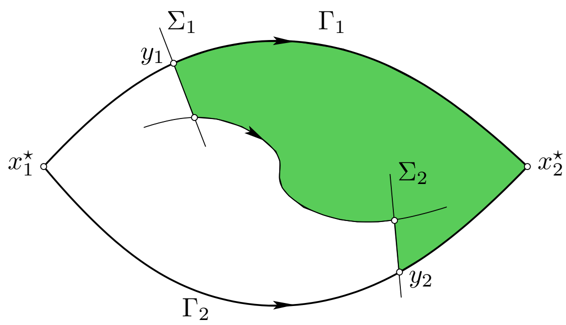

FIGURE 2.5. If the ω-limit set contained two saddle connections Γ_1, Γ_2, admitting the same equilibria as α- and ω-limit sets, the shaded set would be invariant.

3. **Let $x_1^\star$ and $x_2^\star$ be distinct equilibrium points contained in $\omega(x)$. There exists at most one orbit $\Gamma \in \omega(x)$ such that $\alpha(y) = x_1^\star$ and $\omega(y) = x_2^\star$ for all $y \in \Gamma$.**
   Assume there exist two orbits Γ_1, Γ_2 with this property. Choose points $y_1 \in \Gamma_1$ and $y_2 \in \Gamma_2$ and construct transverse arcs Σ_1, Σ_2 through these points. Since $\Gamma_1, \Gamma_2 \in \omega(x)$, there exist times $t_2 > t_1 > 0$ such that $\varphi_{t_j}(x) \in \Sigma_j$ for $j = 1, 2$. But this defines an invariant region (Fig. 2.5) that does not contain Γ_1, Γ_2, a contradiction.
4. **If $\omega(x)$ contains only equilibrium points, then it must consist of a unique equilibrium.**
   This follows from the assumption that there are only finitely many equilibria and $\omega(x)$ is connected.
5. **Assume $\omega(x)$ contains both equilibrium points and points that are not equilibria. Then any point in $\omega(x)$ admits equilibria as α- and ω-limit sets.**
   Let y be a point in $\omega(x)$ which is not an equilibrium. If $\alpha(y)$ and $\omega(y)$ were not equilibria, they must be periodic orbits by point 2. But this is impossible since $\omega(x)$ is connected and contains equilibria. □

One of the useful properties of limit sets is that many points share the same limit set, and thus the number of these sets is much smaller than the number of orbits.

Definition 2.1.18. *Let $\omega_0 = \omega(x)$ be the ω-limit set of a point $x \in \mathcal{D}$. The* basin of attraction *of ω_0 is the set $\mathcal{A}(\omega_0) = \{y \in \mathcal{D} \colon \omega(y) = \omega_0\}$.*

We leave it as an exercise to show that if $\mathcal{S}$ is a positively invariant set containing ω_0, then the sets $\mathcal{A}(\omega_0) \cap \mathcal{S}$, $\mathcal{S} \setminus \mathcal{A}(\omega_0)$ and $\partial\mathcal{A}(\omega_0) \cap \mathcal{S}$ are positively invariant. As a consequence, $\mathcal{S}$ can be decomposed into a union of disjoint basins of attraction, whose boundaries consist of periodic orbits and stable manifolds of equilibria (see Fig. 2.6).

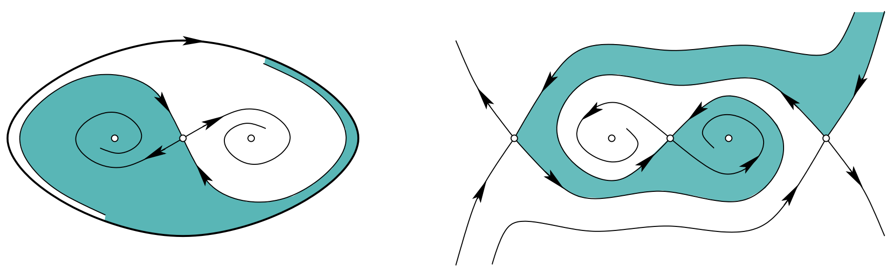

FIGURE 2.6. Examples of basins of attraction.

2.2 Structurally Stable Vector Fields

Loosely speaking, a vector field f is *structurally stable* if any sufficiently small perturbation does not change the qualitative dynamics of $\dot{x} = f(x)$.

Obviously, to make this a precise definition, we need to specify what we mean by "small perturbation" and "same qualitative dynamics". After having done this, we will state a general result, mainly due to Peixoto, which gives necessary and sufficient conditions for a vector field f to be structurally stable.

2.2.1 Definition of Structural Stability

We start by defining what we mean by a small perturbation. To avoid certain technical difficulties, we will work on a *compact* phase space $\mathcal{D} \subset \mathbb{R}^2$.

We denote by $\mathcal{X}^k(\mathcal{D})$ the space of $\mathcal{C}^k$ vector fields on $\mathcal{D}$, pointing inwards on the boundary of $\mathcal{D}$, so that $\mathcal{D}$ is positively invariant. On $\mathcal{X}^k(\mathcal{D})$, we introduce the norm

$$\|f\|_{\mathcal{C}^k} := \sup_{x\in\mathcal{D}} \max_{j=1,2} \max_{\substack{0\leqslant p_1+p_2\leqslant k \\ p_1,p_2\geqslant 0}} \left| \frac{\partial^{p_1+p_2} f_j}{\partial x_1^{p_1} \partial x_2^{p_2}} \right|, \tag{2.2.1}$$

called the $\mathcal{C}^k$*-norm.* Note that $\|\cdot\|_{\mathcal{C}^0}$ is simply the sup norm. $\|f\|_{\mathcal{C}^k}$ is small if the sup norms of f_1, f_2 and all their derivatives up to order k are small. The resulting topology in $\mathcal{X}^k(\mathcal{D})$ is the $\mathcal{C}^k$*-topology.* Note that $(\mathcal{X}^k(\mathcal{D}), \|\cdot\|_{\mathcal{C}^k})$ is a Banach space.

We will say that the vector field $g \in \mathcal{X}^k(\mathcal{D})$ is a small perturbation of $f \in \mathcal{X}^k(\mathcal{D})$ if $\|g - f\|_{\mathcal{C}^k}$ is small, for a certain k. Taking $k = 0$ yields a notion of proximity which is too weak (i.e. too large neighbourhoods in the function space $\mathcal{X}^k(\mathcal{D})$). Indeed, consider for instance the one-dimensional example $f(x) = -x$, $g(x,\varepsilon) = -x + \varepsilon\sqrt{|x|}$, $\mathcal{D} = [-1,1]$. Then $\|g - f\|_{\mathcal{C}^0} = \varepsilon$, but for every $\varepsilon > 0$, g has two equilibria at 0 and ε^2, while f has only one equilibrium. To obtain interesting results, we will need to work with $k = 1$ at least, and sometimes with larger k.

We now introduce a way to compare qualitative dynamics.

Definition 2.2.1. *Two vector fields f and $g : \mathcal{D} \to \mathbb{R}^2$ are said to be* topologically equivalent *if there is a homeomorphism $h : \mathcal{D} \to \mathcal{D}$ (i.e. h is continuous, bijective, and has a continuous inverse) taking orbits of f onto orbits of g, and preserving direction of time.*

In other words, if we denote by φ_t and ψ_t the flows of f and g respectively, the vector fields are topologically equivalent if there exists a homeomorphism h and a continuous, monotonously increasing bijection $\tau : \mathbb{R} \to \mathbb{R}$ such that

$$\psi_{\tau(t)}(x) = h \circ \varphi_t \circ h^{-1}(x) \tag{2.2.2}$$

for all $x \in \mathcal{D}$ and all $t \geqslant 0$.

Since h transforms orbits of f into orbits of g, both vector fields also have homeomorphic invariant sets, and in particular the same type of α- and ω-limit sets. Thus, if we understand the qualitative dynamics of f (in particular, asymptotically for $t \to \pm\infty$), we also understand the dynamics of g, even if we don't know the homeomorphism h explicitly.

We can also define *differentiable equivalence* by requiring that h be $\mathcal{C}^1$ (or $\mathcal{C}^k$ for some $k \geqslant 1$). This, however, often turns out to be too strong a requirement to be helpful in classifying vector fields. Consider for instance the linear system

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{pmatrix} x. \tag{2.2.3}$$

The orbits are of the form $x_2 = x_1^{1+\varepsilon}$. For $\varepsilon = 0$, they belong to straight lines through the origin, while for $\varepsilon > 0$, they are tangent to the x_1-axis. Thus, the systems for $\varepsilon = 0$ and $\varepsilon > 0$ are topologically equivalent, but not differentiably equivalent. On the other hand, topological equivalence will only distinguish between sources, sinks and saddles, but not between nodes and foci.

We will choose the following definition of structural stability:

Definition 2.2.2. *The vector field $f \in \mathcal{X}^k(\mathcal{D})$, $k \geqslant 1$, is* structurally stable *if there exists $\varepsilon > 0$ such that every $g \in \mathcal{X}^k(\mathcal{D})$ with $\|g - f\|_{\mathcal{C}^k} < \varepsilon$ is topologically equivalent to f.*

This definition may seem a bit arbitrary. If we only allow perturbations which are small in the $\mathcal{C}^1$-topology, why do we not require differentiable equivalence? The reason has to do with the size of equivalence classes. If we were to allow perturbations which are only small in the $\mathcal{C}^0$-topology, virtually no vector field would be structurally stable, because one can find continuous perturbations which change the orbit structure. On the other hand, differentiable equivalence is such a strong requirement that it would lead to a very large number of relatively small equivalence classes. Definition 2.2.2 turns out to have exactly the right balance to yield a useful result, which we now state.

2.2.2 Peixoto's Theorem

A characterization of structurally stable vector fields was first given by Andronov and Pontrjagin in 1937, and generalized by Peixoto in 1959. We state here a particular version of Peixoto's result.

Theorem 2.2.3 (Peixoto). *A vector field $f \in \mathcal{X}^1(\mathcal{D})$ is structurally stable if and only if it satisfies the following properties:*

1. *f has finitely many equilibrium points, all being hyperbolic;*
2. *f has finitely many periodic orbits, all being hyperbolic;*
3. *f has no saddle connections.*

Moreover, the set of all structurally stable vector fields is dense in $\mathcal{X}^1(\mathcal{D})$.

Note that f is allowed to have no equilibrium or no periodic orbit at all (although, if $\mathcal{D}$ is positively invariant, we know that there exists a non-empty ω-limit set). We will not give a full proof of this result, which is quite involved, but we shall discuss the main ideas of the proof below. First of all, a few remarks are in order:

- The conditions for f to be structurally stable are remarkably simple, even though they are not always easy to check, especially the absence of saddle connections.
- The fact that the structurally stable vector fields are dense is a very strong result. It means that given any structurally *unstable* vector field, one can find an arbitrarily small perturbation which makes it structurally stable (this is no longer true in three dimensions). It also means that a "typical" vector field in $\mathcal{X}^k(\mathcal{D})$ will contain only hyperbolic equilibria and hyperbolic periodic orbits, and no saddle connections.

- This result allows to classify vector fields with respect to topological equivalence. This can be done, roughly speaking, by associating a graph with each vector field: the vertices of the graph correspond to α- and ω-limit sets, and the edges to orbits connecting these limit sets (this is not always sufficient to distinguish between different classes of vector fields, and some extra information has to be added to the graph).
- Peixoto has given various generalizations of this result, to domains $\mathcal{D}$ which are not positively invariant (in that case, one has to impose certain conditions on the behaviour near the boundary), and to general compact two-dimensional manifolds (where an additional condition on the limit sets is required).

Before proceeding to the sketch of the proof, let us recall the implicit function theorem:

Theorem 2.2.4. *Let $\mathcal{N}$ be a neighbourhood of $(x^\star, y^\star)$ in $\mathbb{R}^n \times \mathbb{R}^m$. Let $\Phi : \mathcal{N} \to \mathbb{R}^n$ be of class $\mathcal{C}^k$, $k \geqslant 1$, and satisfy*

$$\Phi(x^\star, y^\star) = 0, \tag{2.2.4}$$

$$\det \frac{\partial \Phi}{\partial x}(x^\star, y^\star) \neq 0. \tag{2.2.5}$$

Then there exists a neighbourhood $\mathcal{U}$ of $y^\star$ in $\mathbb{R}^m$ and a unique function $\varphi \in \mathcal{C}^k(\mathcal{U}, \mathbb{R}^n)$ such that

$$\varphi(y^\star) = x^\star, \tag{2.2.6}$$

$$\Phi(\varphi(y), y) = 0 \qquad \textit{for all } y \in \mathcal{U}. \tag{2.2.7}$$

Some ideas of the proof of Peixoto's Theorem.
To prove that the three given conditions are sufficient for structural stability, we have to prove the following: Let f satisfy Conditions 1.–3., and let $g \in \mathcal{X}^1(\mathcal{D})$ with $\|g - f\|_{\mathcal{C}^1}$ small enough. Then there exists a homeomorphism h taking orbits of f onto orbits of g. It will be convenient to introduce the one-parameter family

$$F(x, \varepsilon) = f(x) + \varepsilon \Big(\frac{g(x) - f(x)}{\varepsilon_0}\Big), \qquad \varepsilon_0 = \|g - f\|_{\mathcal{C}^1},$$

which satisfies $F(x, 0) = f(x)$, $F(x, \varepsilon_0) = g(x)$ and $\|F(\cdot, \varepsilon) - f\|_{\mathcal{C}^1} = \varepsilon$. The main steps of the proof are the following:

1. Let $x^\star$ be an equilibrium point of f. Then we have $F(x^\star, 0) = 0$ by definition, and $\frac{\partial F}{\partial x}(x^\star, 0) = \frac{\partial f}{\partial x}(x^\star)$ is the Jacobian matrix of f at $x^\star$. Since $x^\star$ is hyperbolic by assumption, this matrix has a nonvanishing determinant. Hence the implicit function theorem yields the existence, for sufficiently small ε, of a unique equilibrium point of $F(x, \varepsilon)$ in a neighbourhood of $x^\star$. This equilibrium is also hyperbolic by continuous dependence of the eigenvalues of $\frac{\partial F}{\partial x}$ on ε.
2. Let Γ be a periodic orbit of f. Let Σ be a transverse arc, intersecting Γ at x_0, and let Π_0 be the associated Poincaré map. For sufficiently small ε, Σ is also transverse to $F(x, \varepsilon)$, and a Poincaré map Π_ε can be defined in some subset of Σ. Consider the function
$$\Phi(x, \varepsilon) = \Pi_\varepsilon(x) - x.$$
Then $\Phi(x_0, 0) = 0$ and $\frac{\partial \Phi}{\partial x}(x_0, 0) = \Pi_0'(x_0) - 1 \neq 0$, since Γ is assumed to be hyperbolic. Thus the implicit function theorem can be applied again to show that Π_ε admits a fixed point for sufficiently small ε, which corresponds to a periodic orbit.

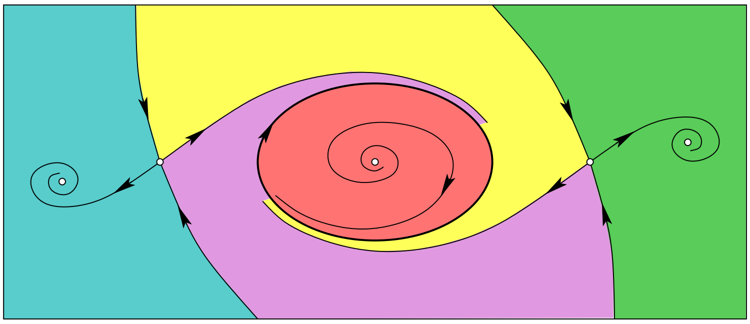

FIGURE 2.7.
Example of a vector field with five canonical regions. Any sufficiently small perturbation will have similar canonical regions, and the homeomorphism h is constructed separately in a neighbourhood of each sink, in a neighbourhood of the periodic orbit, and in each remaining part of a canonical region.

3. Since f has no saddle connections, the Poincaré–Bendixson Theorem shows that f and g have the same α- and ω-limit sets, up to a small deformation. We call *canonical region* a maximal open subset of points in $\mathcal{D}$ sharing the same limit sets (where the boundary $\partial\mathcal{D}$ is considered as an α-limit set). Each canonical region is the intersection of the basin of attraction of an ω-limit set, and the "basin of repulsion" of an α-limit set (Fig. 2.7). Its boundary, called a *separatrix*, consists of equilibria, periodic orbits, and stable/unstable manifolds of saddles. One can show that these manifolds connect the same limit sets even after a small perturbation of f, and thus f and g have homeomorphic canonical regions.
4. The last step is to construct the homeomorphism h. This is done separately in neighbourhoods of all limit sets, and in the remaining part of each canonical region, with appropriate matching on the boundaries. In this way, the sufficiency of Conditions 1.–3. has been proved.

To prove that the given conditions are also necessary, it suffices to prove that if any of them is violated, then one can find $g \in \mathcal{X}^1(\mathcal{D})$ with $\|g - f\|_{\mathcal{C}^1}$ arbitrarily small, such that f and g are not topologically equivalent. We will do this in more detail in the next section. The basic idea is that nonhyperbolic equilibria lead to local bifurcations: the number of limit sets near such an equilibrium can change under a small perturbation. Also, nonhyperbolic periodic orbits may disappear or duplicate when they are perturbed. And saddle connections can be broken by a small transversal perturbation.

Finally, to show that structurally stable vector fields are dense, one proceeds as follows: Let f be structurally unstable and $\varepsilon > 0$ an arbitrarily small number. Then one constructs a structurally stable vector field g such that $\|f - g\| < \varepsilon$. The construction proceeds by successive small deformations of f, which remove the violated conditions one by one. $\square$

2.3 Singularities of Codimension 1

Having obtained a rather precise characterization of structurally stable vector fields, we would now like to understand the structure of the set $\mathcal{S}_0$ of structurally *unstable* vector fields in $\mathcal{X}^k(\mathcal{D})$ (with k sufficiently large). Why should we want to do this, if "typical" vector fields are structurally stable? There are at least two reasons:

- The set $\mathcal{S}_0$ constitutes boundaries between different equivalence classes of structurally stable vector fields, so that elements in $\mathcal{S}_0$ describe transitions between these classes.
- While typical vector fields will not belong to $\mathcal{S}_0$, it is quite possible that one-parameter families f_λ (i.e., curves in $\mathcal{X}^k(\mathcal{D})$) has members in $\mathcal{S}_0$.

By Peixoto's theorem, a structurally unstable vector field will display at least one of the following properties:

- it has a non-hyperbolic equilibrium;
- it has a non-hyperbolic periodic orbit;
- it has a saddle connection.

At least the first two properties can be expressed as conditions of the form $H(f) = 0$, where $H : \mathcal{X}^k(\mathcal{D}) \to \mathbb{R}$ is, for instance, an eigenvalue of a stability matrix. This suggests that $\mathcal{S}_0$ is composed of "hypersurfaces" in the infinite-dimensional space $\mathcal{X}^k(\mathcal{D})$. To make this idea a bit more precise, let us first consider some finite-dimensional examples:

1. If $H : \mathbb{R}^3 \to \mathbb{R}$ is given by $H(x) = x_1 x_2$, the set $\mathcal{S} = H^{-1}(0)$ is composed of the two planes $x_1 = 0$ and $x_2 = 0$. Most points in $\mathcal{S}$ admit a neighbourhood in which $\mathcal{S}$ is a nice "interface" separating two regions. In the vicinity of the line $x_1 = x_2 = 0$, however, $\mathcal{S}$ separates four regions of $\mathbb{R}^3$. Note that the gradient of H vanishes on this line, and only there.
2. If $H : \mathbb{R}^3 \to \mathbb{R}$ is given by $H(x) = x_1^2 + x_2^2 - x_3^2$, the set $\mathcal{S} = H^{-1}(0)$ is a cone. Near any point of $\mathcal{S}$, $\mathcal{S}$ separates two three-dimensional regions, except at the origin, where three such regions meet. Again, the gradient of H vanishes only at this point.

In the infinite-dimensional case of $\mathcal{X}^k(\mathcal{D})$, we will make the following definition:

Definition 2.3.1. *A set $\mathcal{S} \subset \mathcal{X}^k(\mathcal{D})$ is called a $\mathcal{C}^r$* submanifold of codimension 1 *if there exists an open set $\mathcal{U} \subset \mathcal{X}^k(\mathcal{D})$ and a function $H \in \mathcal{C}^r(\mathcal{U}, \mathbb{R})$ such that* $\mathrm{D}H(f) \neq 0$ *in $\mathcal{U}$ and $\mathcal{S} = \{f \in \mathcal{U} \colon H(f) = 0\}$.*

Here $\mathrm{D}H(f)$ should be understood as the Fréchet derivative, that is, a linear operator such that for all sufficiently small g,

$$\begin{aligned} &H(f+g) = H(f) + \mathrm{D}H(f)g + R(g), \\ &\text{with} \quad \lim_{\|g\|_{\mathcal{C}^k} \to 0} \frac{\|R(g)\|_{\mathcal{C}^k}}{\|g\|_{\mathcal{C}^k}} = 0. \end{aligned} \tag{2.3.1}$$

One can show that the Fréchet derivative is given by the following limit, provided it exists and is continuous in f and g:

$$\mathrm{D}H(f)g = \lim_{\varepsilon \to 0} \frac{H(f + \varepsilon g) - H(f)}{\varepsilon}. \tag{2.3.2}$$

The set $\mathcal{S}_0$ of all structurally unstable vector fields is not a submanifold of codimension 1, because, loosely speaking, it contains points where $\mathrm{D}H$ vanishes. We can, however, single out those points of $\mathcal{S}_0$ in a neighbourhood of which $\mathcal{S}_0$ behaves like such a manifold. A way to do this is to start with the following definition.

Definition 2.3.2. *A structurally unstable vector field $f \in \mathcal{S}_0$ is called* singular of codimension 1 *if there exists a neighbourhood $\mathcal{U}_0$ of f in $\mathcal{S}_0$ (in the induced topology) such that*

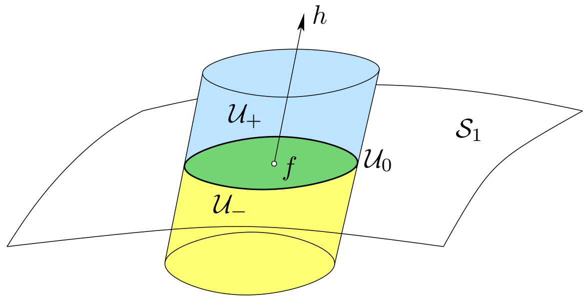

FIGURE 2.8. Schematic representation of $\mathcal{X}^k(\mathcal{D})$ near a singular vector field f of codimension 1. In a small neighbourhood of f, the manifold $\mathcal{S}_1$ of singular vector fields divides $\mathcal{X}^k(\mathcal{D})$ into two components $\mathcal{U}_+$ and $\mathcal{U}_-$, belonging to two different equivalence classes of structurally stable vector fields.

every $g \in \mathcal{U}_0$ is topologically equivalent to f. We will denote by $\mathcal{S}_1$ the set of all singular vector fields of codimension 1.

Then the following result holds:

Theorem 2.3.3. *Let $k \geqslant 4$. Then $\mathcal{S}_1$ is a $\mathcal{C}^{k-1}$ submanifold of codimension 1 of $\mathcal{X}^k(\mathcal{D})$ and is open in $\mathcal{S}_0$ (in the induced topology).*

An important consequence of this result is that if $f \in \mathcal{S}_1$, then there exists a one-parameter family F_λ of vector fields (depending smoothly on λ) such that for all $g \in \mathcal{X}^k(\mathcal{D})$ with $\|g - f\|_{\mathcal{C}^k}$ small enough, g is topologically equivalent to F_λ for some $\lambda \in \mathbb{R}$. The family F_λ is called an *unfolding* of the singularity f.

To see that this is true, we choose a vector field h satisfying the *transversality condition* $\mathrm{D}H(f)h > 0$ (such an h must exist since $\mathrm{D}H(f)$ is not identically zero). Let ε be a sufficiently small number, and define the sets

$$\begin{aligned} \mathcal{U}_0 &= \{g \in \mathcal{S}_1 \colon \|g - f\|_{\mathcal{C}^k} < \varepsilon\} \\ \mathcal{U}_+ &= \{g_\lambda = g_0 + \lambda h \colon g_0 \in \mathcal{U}_0, 0 < \lambda < \varepsilon\} \\ \mathcal{U}_- &= \{g_\lambda = g_0 + \lambda h \colon g_0 \in \mathcal{U}_0, -\varepsilon < \lambda < 0\}. \end{aligned} \tag{2.3.3}$$

Note that by definition, $H = 0$ in $\mathcal{U}_0$. Since, for sufficiently small ε,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} H(g_\lambda) = \mathrm{D}H(g_\lambda)h > 0 \tag{2.3.4}$$

by continuity of $\mathrm{D}H(\cdot)$, we conclude that $H > 0$ in $\mathcal{U}_+$ and $H < 0$ in $\mathcal{U}_-$. Hence all $g \in \mathcal{U}_+$ are structurally stable, and since $\mathcal{U}_+$ is clearly open and connected, they must be topologically equivalent. Similarly, all $g \in \mathcal{U}_-$ are topologically equivalent. This shows that any family g_λ is an unfolding of f, since it has members in $\mathcal{U}_-$, $\mathcal{U}_0$ and $\mathcal{U}_+$.

Unfoldings are mainly used to describe the dynamics in neighbourhoods of nonhyperbolic equilibria, and their perturbations. They are closely related to the so-called *singularity theory* of functions, also known as *catastrophe theory.*

In the sequel, we shall give some ideas of the proof of Theorem 2.3.3 by enumerating all possible singularities of codimension 1, which can be classified into five types. In each case, we will construct a function $H : \mathcal{U} \to \mathbb{R}$, defined in a neighbourhood $\mathcal{U}$ of a given structurally unstable vector field f. This function has the following properties: all $g \in \mathcal{U}_0 = H^{-1}(0)$ are topologically equivalent to f, and the sets $\mathcal{U}_\pm = H^{-1}(\mathbb{R}_\pm)$ belong to two different equivalence classes of structurally stable vector fields.

2.3.1 Saddle–Node Bifurcation of an Equilibrium

A first type of codimension 1 singular vector field occurs when a nonhyperbolic equilibrium point is present, such that the linearization around this point admits 0 as a simple eigenvalue. In appropriate coordinates, the vector field can be written as

$$\begin{aligned}\dot{x}_1 &= g_1(x_1, x_2)\\ \dot{x}_2 &= ax_2 + g_2(x_1, x_2), \qquad\qquad a \neq 0,\end{aligned} \tag{2.3.5}$$

where g_1, g_2 and their derivatives all vanish at the origin. We shall further assume that g_1 and g_2 are of class $\mathcal{C}^2$. The stable manifold theorem (Theorem 2.1.9) admits the following generalization:

Theorem 2.3.4 (Center Manifold Theorem).

- *If $a < 0$, (2.3.5) admits a unique invariant curve W^{s}, tangent to the x_2-axis at the origin, such that $\omega(x) = 0$ for all $x \in W^{\mathrm{s}}$; W^{s} is called the* stable manifold *of* 0.
- *If $a > 0$, (2.3.5) admits a unique invariant curve W^{u}, tangent to the x_2-axis at the origin, such that $\alpha(x) = 0$ for all $x \in W^{\mathrm{u}}$; W^{u} is called the* unstable manifold *of* 0.
- *There exists an invariant curve W^{c}, tangent to the x_1-axis at the origin, called a* center manifold.

The center manifold is not necessarily unique. However, any center manifold can be described, for sufficiently small x_1, by an equation of the form $x_2 = h(x_1)$. Inserting this relation into (2.3.5), we obtain the equation

$$ah(x_1) + g_2(x_1, h(x_1)) = h'(x_1)g_1(x_1, h(x_1)), \tag{2.3.6}$$

which must be satisfied by h. One can show that all solutions of this equation admit the same Taylor expansion around $x_1 = 0$. For our purposes, it will be sufficient to know that $h(x_1) = \mathcal{O}(x_1^2)$ as $x_1 \to 0$.

If $y = x_2 - h(x_1)$ describes the distance of a general solution to the center manifold, one deduces from (2.3.5) and (2.3.6) that y satisfies an equation of the form

$$\dot{y} = G(y, x_1)y, \qquad G(y, x_1) = a + \mathcal{O}(x_1) \tag{2.3.7}$$

for small x_1. Thus W^{c} attracts nearby orbits if $a < 0$ and repels them if $a > 0$, so that there can be no equilibrium points outside W^{c}.

In order to understand the qualitative dynamics near the origin, it is thus sufficient to understand the dynamics on the center manifold. It is governed by the equation

$$\dot{x}_1 = g_1(x_1, h(x_1)) = cx_1^2 + \mathcal{o}(x_1^2), \qquad\qquad c = \frac{1}{2}\frac{\partial^2 g_1}{\partial x_1^2}(0,0). \tag{2.3.8}$$

If $c \neq 0$, which is the typical case, then the orbits on W^{c} will be attracted by the origin from one side, and repelled from the other side (Fig. 2.9a).

Definition 2.3.5. *Let $x^\star$ be a nonhyperbolic equilibrium point of f. Assume that the linearization of f at $x^\star$ has the eigenvalues 0 and $a \neq 0$, and that the equation (2.3.8) on its center manifold satisfies $c \neq 0$. Then $x^\star$ is called an* elementary saddle–node.

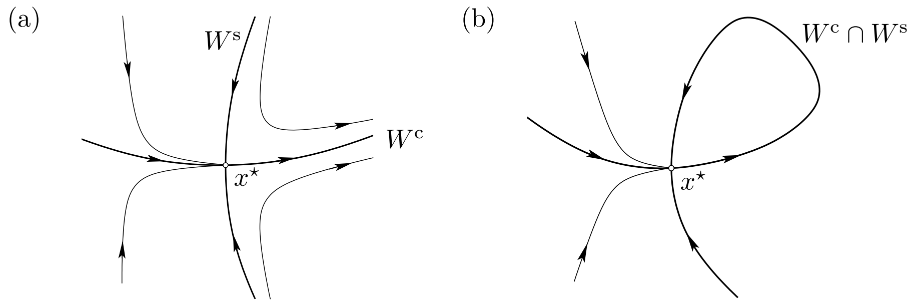

FIGURE 2.9. (a) Local phase portrait near an elementary saddle–node $x^\star$, in a case where $a < 0$. (b) If the center and stable manifolds intersect, then the vector field is not singular of codimension 1, because one can find small perturbations admitting a saddle connection.

Now we would like to examine the effect of a small perturbation of f. Consider to that effect a one-parameter family $f(x, \lambda)$ such that $f(\cdot, 0) = f$, and write the system in the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \lambda), & f_1(x_1, x_2, 0) &= g_1(x_1, x_2), \\ \dot{x}_2 &= f_2(x_1, x_2, \lambda), & f_2(x_1, x_2, 0) &= ax_2 + g_2(x_1, x_2), \\ \dot{\lambda} &= 0. \end{aligned} \tag{2.3.9}$$

The point $(0, 0, 0)$ is a non-hyperbolic equilibrium point of this extended system, with eigenvalues $(0, a, 0)$. The center manifold theorem shows the existence of an invariant manifold, locally described by the equation $x_2 = h(x_1, \lambda)$, which attracts nearby orbits if $a < 0$ and repels them if $a > 0$. The dynamics on the center manifold is described by

$$\dot{x}_1 = f_1(x_1, h(x_1, \lambda), \lambda) =: F(x_1, \lambda), \tag{2.3.10}$$

where $F \in \mathcal{C}^2$ satisfies the relations

$$F(0, 0) = 0, \qquad \frac{\partial F}{\partial x_1}(0, 0) = 0, \qquad \frac{\partial^2 F}{\partial x_1^2}(0, 0) = 2c \neq 0. \tag{2.3.11}$$

The graph of $F(x_1, 0)$ has a quadratic tangency with the x_1-axis at the origin. For small λ, the graph of $F(x_1, \lambda)$ can thus have zero, one or two intersections with the x_1 axis.

Proposition 2.3.6. *For small λ, there exists a differentiable function $H(\lambda)$ such that*

- *if $H(\lambda) > 0$, then $F(x_1, \lambda)$ has no equilibria near $x_1 = 0$;*
- *if $H(\lambda) = 0$, then $F(x_1, \lambda)$ has an isolated non-hyperbolic equilibrium point near $x_1 = 0$;*
- *if $H(\lambda) < 0$, then $F(x_1, \lambda)$ has two hyperbolic equilibrium points of opposite stability near $x_1 = 0$.*

PROOF: Consider the function

$$G(x_1, \lambda) = \frac{\partial F}{\partial x_1}(x_1, \lambda).$$

Then $G(0, 0) = 0$ and $\frac{\partial G}{\partial x_1}(0, 0) = 2c \neq 0$ by (2.3.11). Thus the implicit function theorem shows the existence of a unique differentiable function φ such that $\varphi(0) = 0$ and $G(\varphi(\lambda), \lambda) = 0$ for all sufficiently small λ.

(a)

(b)

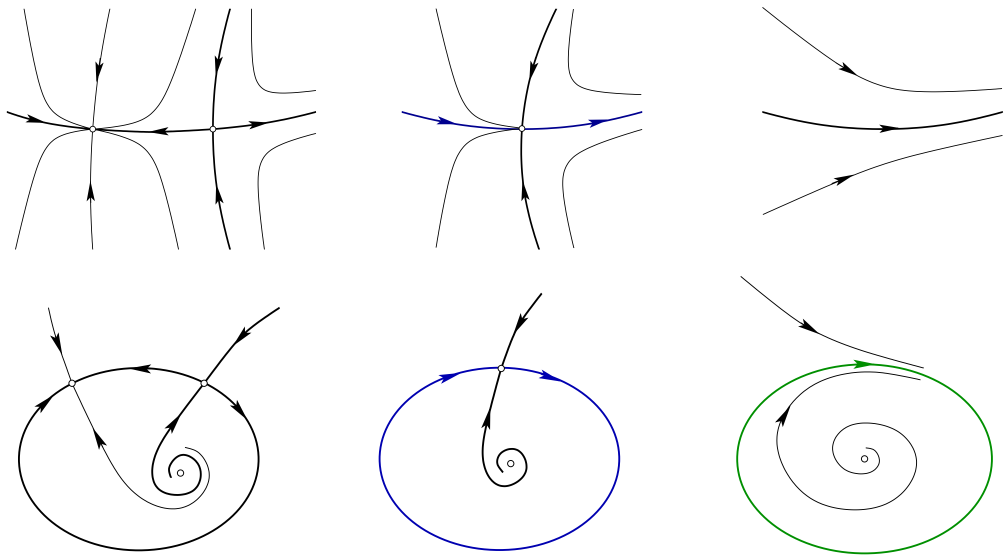

FIGURE 2.10. (a) Unfolding of a saddle–node in a case where the center manifold W^c does not form a loop: the system has a saddle and a node if $H(\lambda) < 0$, an elementary saddle–node if $H(\lambda) = 0$ and there is no equilibrium point if $H(\lambda) > 0$. (b) If the center manifold does form a loop, then a periodic orbit appears for $H(\lambda) > 0$.

Consider now the function $K(y, \lambda) = F(\varphi(\lambda) + y, \lambda)$. Taylor's formula shows that

$$\begin{aligned} K(y,\lambda) &= F(\varphi(\lambda),\lambda) + y^2\big[c + R_1(y,\lambda)\big] \\ \frac{\partial K}{\partial y}(y,\lambda) &= y\big[2c + R_2(y,\lambda)\big], \end{aligned}$$

for some continuous functions R_1, R_2 which vanish at the origin. Let $H(\lambda) = F(\varphi(\lambda), \lambda)/c$. Then $H(0) = 0$, and if λ and y are sufficiently small that $|R_1(y, \lambda)| < c/2$, then

$$H(\lambda) + \frac{1}{2}y^2 \leqslant \frac{K(y,\lambda)}{c} \leqslant H(\lambda) + \frac{3}{2}y^2.$$

Hence $K(y, \lambda)$ vanishes twice, once, or never, depending on the sign of $H(\lambda)$. The expression for $\frac{\partial K}{\partial y}$ shows that K is monotonous in y for small positive or small negative y, which means that there are no other equilibria near the origin. $\square$

Example 2.3.7.

- If $F(x_1, \lambda) = \lambda + x_1^2$, then $H(\lambda) = \lambda$. There are two equilibria for $\lambda < 0$ and no equilibria for $\lambda > 0$. This is the generic case, called the *saddle–node bifurcation*. The family $F(x_1, \lambda)$ constitutes an unfolding of the singularity $f(x_1) = x_1^2$.
- If $F(x_1, \lambda) = \lambda^2 - x_1^2$, then $H(\lambda) = -\lambda^2$. In this case, there are two equilibrium points for all $\lambda \neq 0$, because the family $F(x_1, \lambda)$ is tangent to the manifold $\mathcal{S}_1$. This case is called the *transcritical bifurcation*.
- If $F(x_1, \lambda) = \lambda x_1 - x_1^2$, then $H(\lambda) = -\lambda^2/4$. This is again a transcritical bifurcation.

The existence of the scalar function $H(\lambda)$, which determines entirely the topology of the orbits near the equilibrium point, shows that f is indeed a singularity of codimension 1, in a sufficiently small neighbourhood of the equilibrium. It admits the *local* unfolding

$$\begin{aligned}\dot{x}_1 &= \lambda + x_1^2 \\ \dot{x}_2 &= a x_2.\end{aligned} \tag{2.3.12}$$

For positive H, there are no equilibrium points, while for negative H, the system admits a saddle and a node (Fig. 2.10a).

The *global* behaviour of the perturbed vector field $f(x,\lambda)$ depends on the global behaviour of the center manifold of the saddle–node at $\lambda = 0$. The result is the following.

Theorem 2.3.8. *Assume that $f \in \mathcal{X}^k(\mathcal{D})$, $k \geqslant 2$, admits an elementary saddle–node $x^\star$ as an equilibrium point. Then f is singular of codimension 1 if and only if*

- *all other equilibria and periodic orbits of f are hyperbolic;*
- *f has no saddle connections;*
- *the manifolds $W^{\mathrm{s,u}}(x^\star)$ and $W^{\mathrm{c}}(x^\star)$ have no common point and do not go to a saddle point (Fig. 2.9a).*

If W^{c} does not form a loop, then any small perturbation of f admits, in a small neighbourhood of $x^\star$, either a saddle and a node, or an elementary saddle–node, or no equilibria at all (Fig. 2.10a), all other limit sets being topologically the same.

If W^{c} does form a loop, then any small perturbation of f admits either a saddle and a node near $x^\star$, or an elementary saddle–node near $x^\star$, or a hyperbolic periodic orbit (Fig. 2.10b), all other limit sets being topologically the same.

2.3.2 Hopf Bifurcation

We consider now another type of nonhyperbolic equilibrium point, namely we assume that $f \in \mathcal{C}^3$ vanishes at $x^\star$ and that the linearization of f at $x^\star$ has imaginary eigenvalues $\pm\,\mathrm{i}\,\omega_0$ with $\omega_0 \neq 0$. In appropriate coordinates, we can thus write

$$\begin{aligned}\dot{x}_1 &= -\omega_0 x_2 + g_1(x_1,x_2) \\ \dot{x}_2 &= \omega_0 x_1 + g_2(x_1,x_2),\end{aligned} \tag{2.3.13}$$

where g_1, g_2 are again nonlinear terms, that is, g_1, g_2 and their derivatives vanish for $x_1 = x_2 = 0$. The easiest way to describe the dynamics is to introduce the complex variable $z = x_1 + \mathrm{i}\,x_2$. This amounts to carrying out the change of variables

$$x_1 = \frac{z+\overline{z}}{2} = \operatorname{Re} z, \qquad x_2 = \frac{z-\overline{z}}{2\,\mathrm{i}} = \operatorname{Im} z. \tag{2.3.14}$$

The system (2.3.13) becomes

$$\dot{z} = \mathrm{i}\,\omega_0 z + G(z,\overline{z}), \tag{2.3.15}$$

where G admits a Taylor expansion of the form

$$G(z,\overline{z}) = \sum_{2\leqslant n+m\leqslant 3} G_{nm} z^n \overline{z}^m + \mathcal{O}(|z|^3). \tag{2.3.16}$$

The next step is to simplify the nonlinear term G, using the theory of *normal forms*. Observe to this end that if $h(z,\overline{z})$ is a homogeneous polynomial of degree 2, then the variable $w = z + h(z,\overline{z})$ satisfies the equation

$$\dot{w} = \mathrm{i}\,\omega_0 z + \mathrm{i}\,\omega_0\Big[z\frac{\partial h}{\partial z} - \overline{z}\frac{\partial h}{\partial \overline{z}}\Big] + G(z,\overline{z}) + \frac{\partial h}{\partial z}G(z,\overline{z}) + \frac{\partial h}{\partial \overline{z}}\overline{G}(z,\overline{z}). \tag{2.3.17}$$

Now assume that h satisfies the equation

$$\mathrm{i}\,\omega_0\Big[h - z\frac{\partial h}{\partial z} + \overline{z}\frac{\partial h}{\partial \overline{z}}\Big] = G_2(z,\overline{z}) := \sum_{n+m=2} G_{nm} z^n \overline{z}^m. \tag{2.3.18}$$

Then we see that

$$\dot{w} = \mathrm{i}\,\omega_0 w + G(z,\overline{z}) - G_2(z,\overline{z}) + \frac{\partial h}{\partial z}G(z,\overline{z}) + \frac{\partial h}{\partial \overline{z}}\overline{G}(z,\overline{z}). \tag{2.3.19}$$

All terms of order 2 have disappeared from this equation, while new terms of order 3 have appeared. The right-hand side can be expressed as a function of w and $\overline{w}$ without generating any new terms of order 2. The same procedure can be used to eliminate terms of order 3 as well, by solving an equation analogous to (2.3.18). Let us now consider this equation in more detail. If h is a sum of terms of the form $h_{nm} z^n \overline{z}^m$, we get for each (n,m) the equation

$$\mathrm{i}\,\omega_0(1 - n + m)h_{nm} = G_{nm}. \tag{2.3.20}$$

This equation can be solved if $m \neq n-1$. Hence the only term which cannot be eliminated is $(n,m) = (2,1)$, that is, the term proportional to $|z|^2 z$. We can thus reduce (2.3.15) to the form

$$\dot{w} = \mathrm{i}\,\omega_0 w + c_{21}|w|^2 w + \mathcal{O}(|w|^3), \tag{2.3.21}$$

where the coefficient c_{21} can be expressed as a function of the G_{nm}. If we now introduce polar coordinates $w = r\,\mathrm{e}^{\mathrm{i}\,\varphi}$, substitute in (2.3.21) and take the real and imaginary parts, we end up with the system

$$\begin{aligned} \dot{r} &= \operatorname{Re} c_{21} r^3 + R_1(r,\varphi) \\ \dot{\varphi} &= \omega_0 + \operatorname{Im} c_{21} r^2 + R_2(r,\varphi), \end{aligned} \tag{2.3.22}$$

where $R_1 = \mathcal{O}(r^3)$ and $R_2 = \mathcal{O}(r^2)$. We conclude that if $\operatorname{Re} c_{21} < 0$, solutions starting sufficiently close to the origin will spiral slowly towards $(0,0)$ (r decreases like $1/\sqrt{t}$) (Fig. 2.11). If $\operatorname{Re} c_{21} > 0$, solutions slowly spiral outwards.

Definition 2.3.9. *Assume $x^\star$ is an equilibrium point such that the linearization of f at $x^\star$ admits imaginary eigenvalues $\pm\mathrm{i}\,\omega_0$, $\omega_0 \neq 0$, and such that the normal form* (2.3.22) *satisfies* $\operatorname{Re} c_{21} \neq 0$. *Then $x^\star$ is called an* elementary composed focus.

Let us now consider perturbations of the form $\dot{x} = f(x,\lambda)$ of (2.3.13). Observe that since $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0)$ has no vanishing eigenvalue, the implicit function theorem can be applied to show the existence of a unique equilibrium $x^\star(\lambda)$ near 0 for small λ. The linearization at $x^\star(\lambda)$ has eigenvalues $a(\lambda) \pm \mathrm{i}\,\omega(\lambda)$, where $a(0) = 0$ and $\omega(0) = \omega_0$. In appropriate coordinates, we thus have

$$\begin{aligned} \dot{x}_1 &= a(\lambda)x_1 - \omega(\lambda)x_2 + g_1(x_1,x_2,\lambda) \\ \dot{x}_2 &= \omega(\lambda)x_1 + a(\lambda)x_2 + g_2(x_1,x_2,\lambda). \end{aligned} \tag{2.3.23}$$

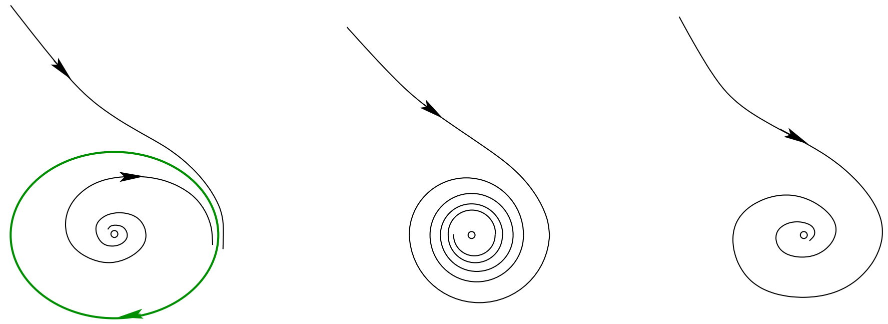

FIGURE 2.11. The unfolding of an elementary composed focus is a Poincaré–Andronov–Hopf bifurcation. Depending on the sign of $a'(0)$, the direction of arrows may be reversed.

The normal form can be computed in the same way, and becomes in polar coordinates

$$\begin{aligned}\dot{r} &= a(\lambda)r + \operatorname{Re} c_{21}(\lambda)r^3 + R_1(r,\varphi,\lambda)\\ \dot{\varphi} &= \omega(\lambda) + \operatorname{Im} c_{21}(\lambda)r^2 + R_2(r,\varphi,\lambda).\end{aligned} \tag{2.3.24}$$

Then the following result holds:

Theorem 2.3.10. *Assume that* $\operatorname{Re} c_{21}(0) \neq 0$ *and let* $H(\lambda) = a(\lambda)/\operatorname{Re} c_{21}(0)$*. Then for sufficiently small* λ*,*

- *if* $H(\lambda) > 0$*,* f *has an isolated hyperbolic equilibrium near* 0*;*
- *if* $H(\lambda) = 0$*,* f *has an elementary composed focus near* 0*;*
- *if* $H(\lambda) < 0$*,* f *has a hyperbolic equilibrium and a hyperbolic periodic orbit near* 0*, with opposite stability.*

The proof should be clear if R_1 is identically zero. If not, this result can be proved by computing $\mathrm{d}r/\mathrm{d}\varphi$ and examining the properties of the Poincaré map associated with the section $\varphi = 0$.

We conclude that if f admits an elementary composed focus, then f is singular of codimension 1 (provided all other limit sets are hyperbolic and there are no saddle connections). The family of equations $\dot{z} = (\lambda + \mathrm{i}\,\omega_0)z + c_{21}|z|^2 z$ is a local unfolding of the singularity.

2.3.3 Saddle–Node Bifurcation of a Periodic Orbit

A third class of structurally unstable vector fields consists of the vector fields which admit a non-hyperbolic periodic orbit Γ. Recall that if Π is the Poincaré map associated with a transverse section Σ through $x^\star \in \Gamma$, then Γ is non-hyperbolic if $\Pi'(x^\star) = 1$, which means that the graph of $\Pi(x)$ is tangent to the diagonal at $x^\star$.

Definition 2.3.11. *A periodic orbit* Γ *is called* quasi-hyperbolic *if its Poincaré map satisfies* $\Pi'(x^\star) = 1$ *and* $\Pi''(x^\star) \neq 0$*.*

The condition $\Pi''(x^\star) \neq 0$ means that the graph of $\Pi(x)$ has a quadratic tangency with the diagonal. Hence, the orbit Γ attracts other orbits from one side, and repels them from the other side (Fig. 2.12).

Let us now consider a one-parameter family of perturbations $f(x,\lambda)$ of the vector field. For sufficiently small λ, $f(x,\lambda)$ will still be transverse to Σ, and it admits a Poincaré

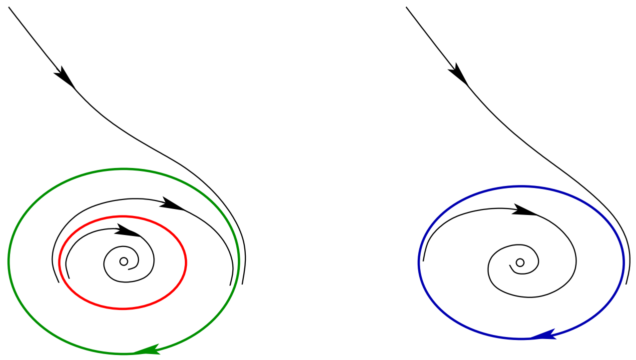

FIGURE 2.12. The unfolding of a quasi-hyperbolic periodic orbit Γ is a saddle–node bifurcation of periodic orbits: A small perturbation will have either two hyperbolic periodic orbits, or one quasi-hyperbolic periodic orbit, or no periodic orbit near Γ. The focus in the center of the pictures does not take part in the bifurcation, and can be replaced by a more complicated limit set.

map $\Pi(x, \lambda)$ defined in some neighbourhood of $(x^\star, 0)$. Periodic orbits correspond to fixed points of Π, and are thus solutions of the equation $\Pi(x, \lambda) - x = 0$.

Note that $\Pi(x, 0) - x$ is tangent to the x-axis at $x = x^\star$. This situation is very reminiscent of the saddle–node bifurcation of equilibrium points, see (2.3.10) and (2.3.11). Indeed, Proposition 2.3.6 applied to $\Pi(x, \lambda) - x = 0$ yields the existence of a function $H(\lambda)$ which controls the number of fixed points of Π.

Theorem 2.3.12. *Let $f(x, 0)$ admit a quasi-hyperbolic periodic orbit Γ. Then there is a function $H(\lambda)$ such that, for sufficiently small λ,*

- *if $H(\lambda) > 0$, then $f(x, \lambda)$ admits no periodic orbit near Γ;*
- *if $H(\lambda) = 0$, then $f(x, \lambda)$ admits a quasi-hyperbolic periodic orbit near Γ ;*
- *if $H(\lambda) < 0$, then $f(x, \lambda)$ admits two hyperbolic periodic orbits of opposite stability near Γ.*

Moreover, $f(x, 0)$ is singular of codimension 1 if and only if there do not exist two invariant manifolds of saddles which approach Γ, one for $t \to \infty$, the other for $t \to -\infty$.

The reason for this last condition is that if such manifolds exist, then one can construct perturbations which admit a saddle connection, and are thus neither structurally stable, nor topologically equivalent to f (Fig. 2.13).

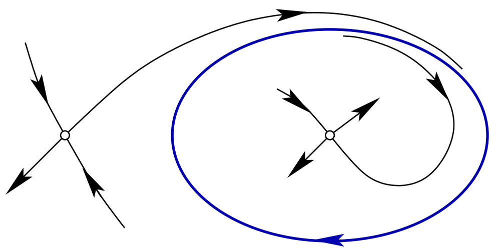

FIGURE 2.13. Partial phase portrait near a quasi-hyperbolic periodic orbit which is not a codimension 1 singularity: a small perturbation may admit a saddle connection.

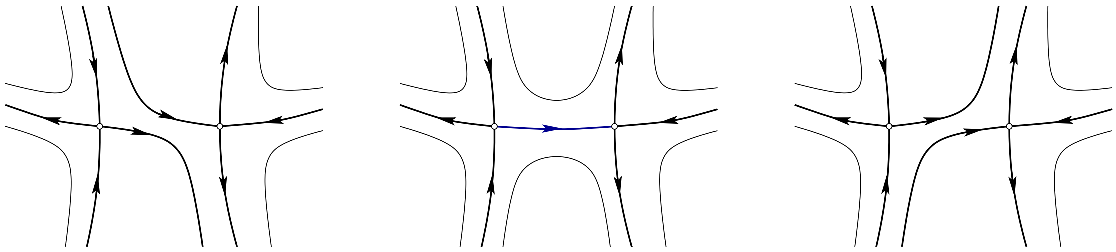

FIGURE 2.14. Breaking a heteroclinic saddle connection.

2.3.4 Global Bifurcations

The remaining singular vector fields of codimension 1 are those which admit a saddle connection. Such saddle connections are indeed very sensitive to small perturbations.

Consider first the case of a heteroclinic saddle connection between two saddle points $x_1^\star$ and $x_2^\star$. To simplify the notations, let us choose coordinates in such a way that $x_1^\star = (0,0)$ and $x_2^\star = (1,0)$, and that the saddle connection belongs to the x_1-axis. We write the perturbed system as

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,x_2) + \lambda g_1(x_1,x_2,\lambda)\\ \dot{x}_2 &= f_2(x_1,x_2) + \lambda g_2(x_1,x_2,\lambda),\end{aligned} \tag{2.3.25}$$

where, in particular, $f_2(x_1,0) = 0$ for all $x_1 \in [0,1]$. We want to find a condition for this system to admit a saddle connection for $\lambda \neq 0$. Note first that since $x_1^\star$ and $x_2^\star$ are hyperbolic, the perturbed system (2.3.25) still admits two saddles near $x_1^\star$ and $x_2^\star$ for small λ. Carrying out an affine coordinate transformation, we may assume that they are still located at $(0,0)$ and $(1,0)$.

Assume that the saddle connection has an equation of the form $x_2 = \lambda h_1(x_1) + \mathcal{O}(\lambda^2)$, with $h_1(0) = h_1(1) = 0$ (such an expansion can be proved to hold for sufficiently small λ). By Taylor's formula, we find that on this curve,

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,0) + \mathcal{O}(\lambda)\\ \dot{x}_2 &= \lambda\frac{\partial f_2}{\partial x_2}(x_1,0)h_1(x_1) + \lambda g_2(x_1,0,0) + \mathcal{O}(\lambda^2).\end{aligned} \tag{2.3.26}$$

Thus h_1 must satisfy the linear equation

$$h_1'(x_1) = \lim_{\lambda\to 0}\frac{\dot{x}_2}{\lambda\dot{x}_1} = \frac{1}{f_1(x_1,0)}\biggl[\frac{\partial f_2}{\partial x_2}(x_1,0)h_1(x_1) + g_2(x_1,0,0)\biggr], \tag{2.3.27}$$

which admits the solution (using $h_1(0) = 0$ as initial value)

$$h_1(x_1) = \int_0^{x_1} \exp\biggl\{\int_y^{x_1} \frac{1}{f_1(z,0)}\frac{\partial f_2}{\partial x_2}(z,0)\,\mathrm{d}z\biggr\}\frac{g_2(y,0,0)}{f_1(y,0)}\,\mathrm{d}y. \tag{2.3.28}$$

This relation can also be written in the more symmetric form

$$h_1(x_1) = \frac{1}{f_1(x_1,0)}\int_0^{x_1} \exp\biggl\{\int_y^{x_1} \frac{1}{f_1}\Bigl[\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}\Bigr](z,0)\,\mathrm{d}z\biggr\}g_2(y,0,0)\,\mathrm{d}y. \tag{2.3.29}$$

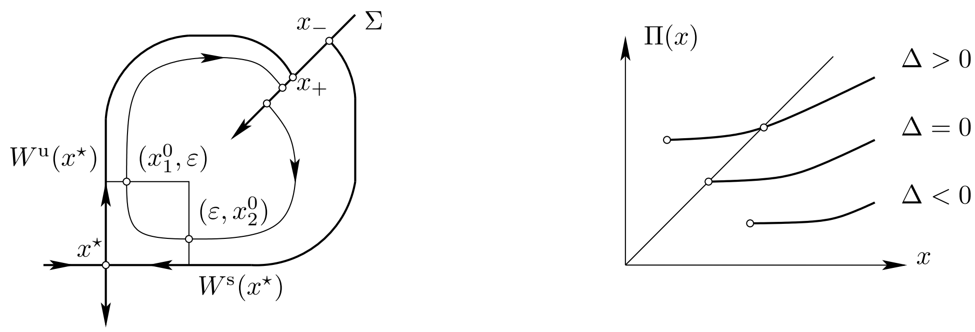

FIGURE 2.15. (a) Definition of the first return map Π associated with a section Σ transverse to the stable and unstable manifolds of $x^\star$. (b) Behaviour of the first return map in a case where $|a_1| > a_2$ for different signs of the splitting $\Delta = x_+ - x_-$. Π admits a fixed point when the splitting is positive, which corresponds to a periodic orbit. If $|a_1| < a_2$, then Π admits a fixed point when the splitting is negative.

Note that only the last factor in the integrand depends on the perturbation term g. Here we have only computed the first order term of $h(x_1, \lambda) = \lambda h_1(x_1) + \mathcal{O}(\lambda^2)$. Including higher order terms, the requirement $h(1, \lambda) = 0$ provides us with a codimension 1 condition of the form $H(\lambda) = 0$. This kind of integral condition is closely related to the method of *Melnikov functions*, which allows to estimate the splitting between manifolds.

If the condition $h(1, \lambda) = 0$ is not satisfied, then the unstable manifold of $x_1^\star$ and the stable manifold of $x_2^\star$ will avoid each other, and the sign of h will determine which manifold lies above the other (Fig. 2.14). (Strictly speaking, $W^{\mathrm{u}}(x_1^\star)$ may not be the graph of a function when x_1 approaches 1, and one would rather compute estimations of $W^{\mathrm{u}}(x_1^\star)$ for $0 < x_1 < 1/2$ and of $W^{\mathrm{s}}(x_2^\star)$ for $1/2 < x < 1$, and determine their splitting at 1/2.)

Let us finally consider the other kind of saddle connection, also called a *homoclinic loop* Γ. In order to describe the flow near Γ, let us introduce a transverse arc Σ through a point $x_0 \in \Gamma$. We order the points on Σ in the inward direction (Fig. 2.15a). Orbits starting slightly "above" x_0 with respect to this ordering will return to Σ, defining a first return map Π for small $x > x_0$.

For small $\lambda \neq 0$, the stable and unstable manifold of $x^\star$ will intersect Σ at points $x_-(\lambda)$ and $x_+(\lambda)$, and their distance $\Delta(\lambda) = x_+(\lambda) - x_-(\lambda)$ may be estimated as in the case of a heteroclinic saddle connection. A first return map $\Pi(x, \lambda)$ can be defined for sufficiently small $x > x_-$ (Fig. 2.15a).

Our aim will now be to determine the behaviour of Π in the limit $x \to x_-$. One can show that in this limit, the first return map depends essentially on the behaviour of the orbits as they pass close to $x^\star$. Assume that the linearization of f at $x^\star$ has eigenvalues $a_1 < 0 < a_2$, and choose a coordinate system where this linearization is diagonal. We consider an orbit connecting the points (ε, x_2^0) and (x_1^0, ε) with $0 < x_2^0 < \varepsilon \ll 1$, and would like to determine x_1^0 as a function of x_2^0. In a linear approximation, we have

$$\begin{aligned} \dot{x}_1 &= -|a_1| x_1 & &\Rightarrow & x_1(t) &= \mathrm{e}^{-|a_1|t}\,\varepsilon \\ \dot{x}_2 &= a_2 x_2 & &\Rightarrow & x_2(t) &= \mathrm{e}^{a_2 t}\,x_2^0. \end{aligned} \tag{2.3.30}$$

Requiring that $x_2(t) = \varepsilon$ and eliminating t, we obtain

$$\frac{x_1^0}{x_2^0} = \frac{\mathrm{e}^{-|a_1|\log(\varepsilon/x_2^0)/a_2}\,\varepsilon}{x_2^0} = \left(\frac{\varepsilon}{x_2^0}\right)^{1-|a_1|/a_2}. \tag{2.3.31}$$

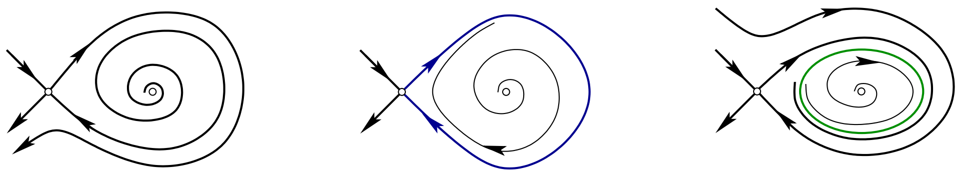

FIGURE 2.16. Unfolding an elementary homoclinic loop Γ: any small perturbation will either admit a hyperbolic periodic orbit close to Γ, or an elementary homoclinic loop, or no invariant set near Γ.

We conclude that

$$\lim_{x_2^0 \to 0} \frac{x_1^0}{x_2^0} = \begin{cases} 0 & \text{if } |a_1| > a_2 \\ \infty & \text{if } |a_1| < a_2. \end{cases} \tag{2.3.32}$$

By controlling the error terms a bit more carefully, one can show that $\lim_{x \to x_-} \Pi'(x)$ has the same behaviour, as long as $|a_1|/a_2$ is bounded away from 1.

Definition 2.3.13. *Let $x^\star$ be a saddle point with eigenvalues $a_1 < 0 < a_2$ satisfying $a_1 + a_2 \neq 0$. An orbit Γ such that $\omega(y) = \alpha(y) = x^\star$ for all $y \in \Gamma$ is called an* elementary homoclinic loop.

In summary, we know that as $x \to x_-$, the first return map of an elementary homoclinic loop satisfies

$$\begin{aligned} \lim_{x \to x_-} \Pi(x) &= x_- + \Delta(\lambda) \\ \lim_{x \to x_-} \Pi'(x) &= \begin{cases} 0 & \text{if } |a_1| > a_2 \\ \infty & \text{if } |a_1| < a_2. \end{cases} \end{aligned} \tag{2.3.33}$$

The shape of Π for different values of Δ is sketched in Fig. 2.15b, in a case where $|a_1| > a_2$. We see that if $\Delta > 0$, Π must admit a fixed point, which corresponds to a periodic orbit (Fig. 2.16). If $|a_1| < a_2$, there is a periodic orbit for $\Delta < 0$.

Theorem 2.3.14. *Let f admit an elementary homoclinic loop Γ. Given a one-parameter family of perturbations $f(x, \lambda)$, there exists a function $H(\lambda) = \Delta(\lambda) \log(|a_1|/a_2)$ (where Δ describes the splitting of stable and unstable manifolds) such that*

- *if $H(\lambda) < 0$, then there are no invariant sets near Γ;*
- *if $H(\lambda) = 0$, then $f(x, \lambda)$ admits an elementary homoclinic loop near Γ;*
- *if $H(\lambda) > 0$, then $f(x, \lambda)$ admits a hyperbolic periodic orbit near Γ.*

If, moreover, all other equilibria and periodic orbits of f are hyperbolic, there are no saddle connections, and no stable or unstable manifold of another saddle point tends to Γ, then f is a singularity of codimension 1.

This completes the list of all structurally unstable vector fields of codimension 1: $f \in \mathcal{S}_1$ if and only if it satisfies the hypotheses of Peixoto's theorem, except for one limit set, chosen from the following list:

1. an elementary saddle-node (satisfying the hypotheses of Theorem 2.3.8);
2. an elementary composed focus;
3. a quasi-hyperbolic periodic orbit (satisfying the hypotheses of Theorem 2.3.12);
4. a heteroclinic saddle connection;
5. an elementary homoclinic loop (satisfying the hypotheses of Theorem 2.3.14).

2.4 Local Bifurcations of Codimension 2

Once we have understood the set $\mathcal{S}_1$ of singular vector fields of codimension 1, the next step would be to characterize the remaining structurally unstable vector fields in $\mathcal{S}_0 \setminus \mathcal{S}_1$. In analogy with the codimension 1 case, we can define a set $\mathcal{S}_2 \subset \mathcal{S}_0 \setminus \mathcal{S}_1$ of singular vector fields of codimension 2 in two equivalent ways:

1. a vector field $f \in \mathcal{S}_0 \setminus \mathcal{S}_1$ is singular of codimension 2 if any sufficiently small perturbation g of f which still belongs to $\mathcal{S}_0 \setminus \mathcal{S}_1$ is topologically equivalent to f;
2. a vector field $f \in \mathcal{S}_0 \setminus \mathcal{S}_1$ is singular of codimension 2 if there exists a two-parameter family $F_{\lambda_1\lambda_2}$ of vector fields such that *any* sufficiently small perturbation of f is topologically equivalent to a member of the family $F_{\lambda_1\lambda_2}$.

More generally, we could define singular vector fields of higher codimension, and thus decompose $\mathcal{S}_0$ into a disjoint union of components

$$\mathcal{S}_0 = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \dots \cup \mathcal{S}_k \cup \dots \tag{2.4.1}$$

The classification of vector fields in $\mathcal{S}_k$ becomes, however, increasingly difficult as k grows, and is not even complete in the case $k = 2$. The discussion in the previous section shows that the following vector fields are candidates for belonging to $\mathcal{S}_2$:

1. vector fields containing *two* non-hyperbolic equilibrium points, periodic orbits or saddle connections (or any combination of them);
2. vector fields with a non-hyperbolic equilibrium or periodic orbit having more than one vanishing term in their Taylor series;
3. vector fields with a non-hyperbolic equilibrium point having a double zero eigenvalue;
4. structurally unstable vector fields which may admit a saddle connection after a small perturbation (as in Fig. 2.9b and Fig. 2.13).

The first type is not very interesting, and the fourth type is very hard to analyse. We will discuss below the simplest examples of types 2. and 3.

2.4.1 Pitchfork Bifurcation

Let $f \in \mathcal{C}^3$ have a non-hyperbolic equilibrium point $x^\star$, such that the Jacobian matrix $\frac{\partial f}{\partial x}(x^\star)$ admits the eigenvalues 0 and $a \neq 0$. Unlike in the case of the saddle–node bifurcation, let us assume that the dynamics on the center manifold is governed by the equation $\dot{x_1} = F(x_1)$ with

$$F(0) = \frac{\partial F}{\partial x}(0) = \frac{\partial^2 F}{\partial x^2}(0) = 0, \qquad \frac{\partial^3 F}{\partial x^3}(0) = 6c \neq 0. \tag{2.4.2}$$

Then $\dot{x_1} = cx_1^3 + \mathcal{O}(x_1^3)$, and thus $x^\star$ will attract nearby solutions on the center manifold from both sides if $c < 0$, and repel them if $c > 0$, but at a much slower rate than in the hyperbolic case.

A perturbation $f(x, \lambda)$ of f will admit a center manifold on which the dynamics is governed by an equation $\dot{x_1} = F(x_1, \lambda)$. We claim that for sufficiently small λ and x_1, this system is topologically equivalent to a member of the 2-parameter family

$$\dot{x}_1 = F_{\lambda_1\lambda_2}(x_1) = \lambda_1 + \lambda_2 x_1 \pm x_1^3, \tag{2.4.3}$$

where the signs $\pm$ correspond to the cases $c > 0$ or $c < 0$. Let us focus on the $-$ case, since the other case can be obtained by inverting the direction of time. As we saw in Example 1.1.2, the dynamics for $(\lambda_1, \lambda_2) \neq (0, 0)$ can be of three types (Fig. 1.1):

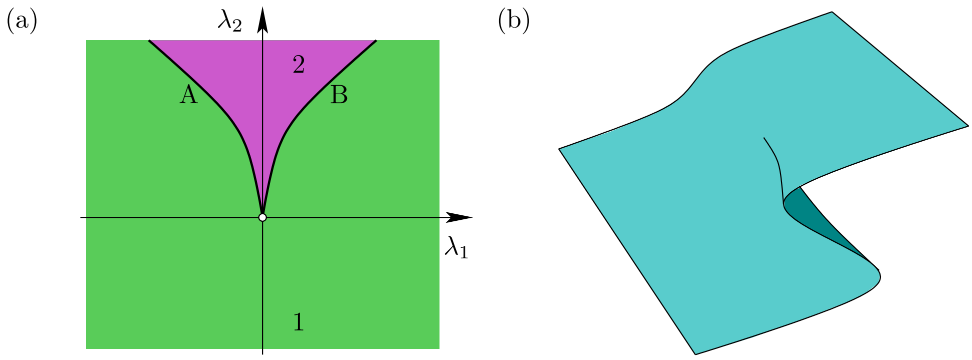

FIGURE 2.17. (a) Bifurcation diagram of the family $\dot{x}_1 = \lambda_1 + \lambda_2 x_1 - x_1^3$. The lines A and B correspond to saddle–node bifurcations. (b) The surface $\lambda_1 + \lambda_2 x_1 - x_1^3 = 0$ has a fold in the space $(\lambda_1, \lambda_2, x_1)$.

1. if $4\lambda_2^3 < 27\lambda_1^2$, $F_{\lambda_1\lambda_2}$ has exactly one equilibrium point, which is hyperbolic and stable;
2. if $4\lambda_2^3 > 27\lambda_1^2$, $F_{\lambda_1\lambda_2}$ has three equilibrium points, all being hyperbolic, and two of them stable while the third one is unstable;
3. if $4\lambda_2^3 = 27\lambda_1^2$, $F_{\lambda_1\lambda_2}$ has two equilibrium points; one of them is hyperbolic and stable, and the other one is nonhyperbolic with a quadratic tangency to the x_1-axis, corresponding to an elementary saddle–node of the associated two-dimensional system.

The singular lines $4\lambda_2^3 = 27\lambda_1^2$ form a cusp in the (λ_1, λ_2)-plane (Fig. 2.17). A one-parameter family of vector fields crossing one of these curves away from the cusp undergoes a saddle–node bifurcation. Crossing the origin from one side of the cusp to the other one changes the number of equilibria from 1 to 3 and is called a *pitchfork bifurcation.*

Proposition 2.4.1. *The two-parameter family* (2.4.3) *is a local unfolding of the singular vector field whose restriction to the center manifold satisfies* (2.4.2)*.*

PROOF: The function

$$G(x_1, \lambda) = \frac{\partial F}{\partial x_1}(x_1, \lambda)$$

satisfies the hypotheses of F in Proposition 2.3.6, that we used in the case of the saddle–node bifurcation. There we showed the existence of a function $H(\lambda)$ such that, in a neighbourhood of $x_1 = 0$, G vanishes twice if $H < 0$, once if $H = 0$ and never if $H > 0$. It follows that F has two extrema if $H < 0$, is monotonous with an inflection point if $H = 0$, and is strictly monotonous if $H > 0$. This shows in particular that H plays the role of λ_2 in the unfolding (2.4.3).

Now let $\varphi(\lambda)$ be the extremum of G (see Proposition 2.3.6), i.e. the inflection point of F. Then the Taylor expansion of F around $\varphi(\lambda)$ can be written as

$$\frac{1}{c}F(\varphi(\lambda) + y, \lambda) = \frac{1}{c}F(\varphi(\lambda), \lambda) + \frac{1}{c}\frac{\partial F}{\partial x_1}(\varphi(\lambda), \lambda)y + y^3\big[1 + R(y, \lambda)\big],$$

where $R(y, \lambda)$ is a continuous function with $R(0,0) = 0$. The coefficient of y is exactly $H(\lambda) = \lambda_2$. Since $F(\varphi(\lambda), \lambda)/c$ controls the vertical position of the graph of F, it plays the same rôle as λ_1 (one can show that F has the same behaviour as $F_{\lambda_1\lambda_2}$ for a λ_1 of the form $c^{-1}F(\varphi(\lambda), \lambda)[1 + \rho(\lambda)]$, with $\rho(\lambda) \to 0$ as $\lambda \to 0$). $\square$

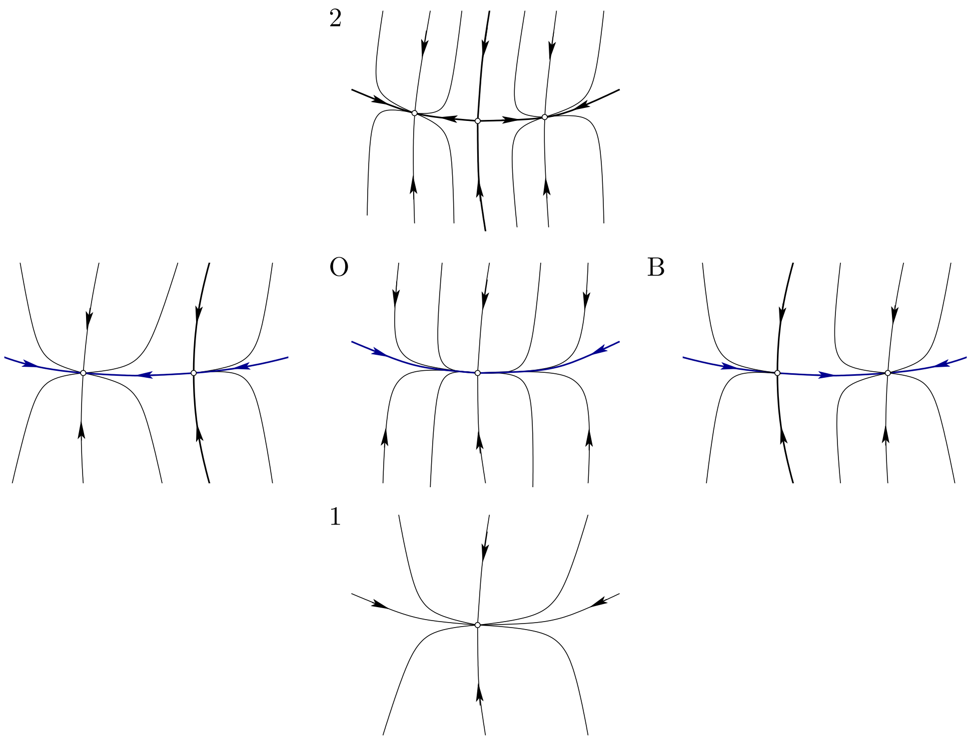

FIGURE 2.18. Unfolding a singular vector field (O) equivalent to $(-x_1^3, -x_2)$. Structurally stable perturbations have either one hyperbolic equilibrium (1) or three hyperbolic equilibria (2). They are separated by singular vector fields containing an elementary saddle–node and a hyperbolic equilibrium (A,B).

2.4.2 Takens–Bogdanov Bifurcation

Consider now the case where f has a non-hyperbolic equilibrium point $x^\star$, such that the linearization of f at $x^\star$ admits 0 as a double eigenvalue. Then $\frac{\partial f}{\partial x}(x^\star)$ can admit one of the two Jordan canonical forms

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4.4}$$

The second case does not correspond to a codimension 2 bifurcation. This can be roughly understood as follows: Any 2×2 matrix is equivalent to a triangular matrix

$$\begin{pmatrix} \lambda_1 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix}. \tag{2.4.5}$$

None of the systems obtained by setting two among the three λ_j equal to zero is topologically equivalent to the system with all three λ_j equal to zero, and thus two parameters do not suffice to describe all possible perturbations of f.

Consider now the first case. In appropriate coordinates, we can write

$$\begin{aligned} \dot{x}_1 &= x_2 + g_1(x_1, x_2) \\ \dot{x}_2 &= g_2(x_1, x_2), \end{aligned} \tag{2.4.6}$$

where g_1, g_2 and all their first order derivatives vanish at the origin. These nonlinear terms can be simplified by a change of variables, and the general theory describing how to do this is the theory of *normal forms*. We will only illustrate it in this particular case. First, note that the Jacobian of the change of variables $(x_1, x_2) \mapsto (x_1, x_2 + g_1(x_1, x_2))$ is close to unity near $(x_1, x_2) = 0$, and transforms the system into

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= g(x_1, x_2), \end{aligned} \tag{2.4.7}$$

where an easy calculation shows that g has the same properties as g_1 and g_2. Note that this system is equivalent to the second order equation

$$\ddot{x}_1 = g(x_1, \dot{x}_1), \tag{2.4.8}$$

called a *Liénard equation*. As we did in the case of the Hopf bifurcation, we can try to simplify the term g further by introducing the new variable $y_1 = x_1 + h(x_1, \dot{x}_1)$, where h is a homogeneous polynomial of degree 2. Differentiating y_1 and using (2.4.8), we obtain

$$\dot{y}_1 = \dot{x}_1 \Big(1 + \frac{\partial h}{\partial x_1}\Big) + \frac{\partial h}{\partial \dot{x}_1} g(x_1, \dot{x}_1). \tag{2.4.9}$$

If we differentiate this relation again, we obtain several terms, but most of them are of order 3 (or larger):

$$\ddot{y}_1 = g(x_1, \dot{x}_1) + \dot{x}_1^2 \frac{\partial^2 h}{\partial x_1^2} + \text{terms of order 3.} \tag{2.4.10}$$

Now if $h(x_1, \dot{x}_1) = h_1 x_1^2 + h_2 x_1 \dot{x}_1 + h_3 \dot{x}_1^2$, we see that we can choose h_1 in such a way as to eliminate the term proportional to $\dot{x}_1^2$ of g, while h_2 and h_3 have no influence on quadratic terms. The other second order terms of g cannot be eliminated, and are called *resonant*. Expressing the remaining terms as functions of y_1, y_2, we obtain the normal form of (2.4.7) to second order

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= c_1 y_1^2 + c_2 y_1 y_2 + R(y_1, y_2), \end{aligned} \tag{2.4.11}$$

where c_1 and c_2 are Taylor coefficients of g and R is a remainder of order 3. Takens has proved the nontrivial fact that if $c_1 \neq 0$, then the remainder R does not change the topology of the orbits near the origin. If $c_1 \neq 0$ and $c_2 \neq 0$, one can reduce the problem, by scaling y_1, y_2 and t, to one of the cases $c_1 = 1$ and $c_2 = \pm 1$. We shall consider here the case $c_2 = 1$, that is,

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_1^2 + y_1 y_2. \end{aligned} \tag{2.4.12}$$

The case $c_2 = -1$ is obtained by inverting the direction of time (and changing the sign of y_2). The system (2.4.12) is not straightforward to analyse by itself, but one can check that there exist two orbits of the form

$$y_2 = \pm \Big(\frac{2}{3}\Big)^{1/2} y_1^{3/2} [1 + \rho(y_1)], \qquad y_1 > 0, \qquad \rho(y_1) = \mathcal{O}(\sqrt{y_1}), \tag{2.4.13}$$

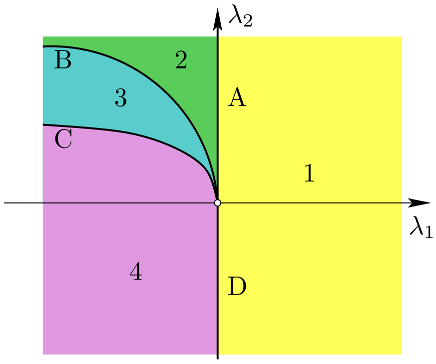

FIGURE 2.19. Bifurcation diagram of the Takens–Bogdanov singularity. The lines A and D correspond to saddle–node bifurcations, B to a Hopf bifurcation, and C to a homoclinic loop bifurcation.

which form a cusp (Fig. 2.20–O). Takens and Bogdanov have shown that the following two-parameter family of vector fields constitutes a local unfolding of the singular vector field (2.4.12):

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= \lambda_1 + \lambda_2 y_2 + y_1^2 + y_1 y_2. \end{aligned} \tag{2.4.14}$$

(Various equivalent unfoldings are found in the literature). Proving this lies beyond the scope of the present course. Let us, however, examine the dynamics of (2.4.14) as a function of λ_1 and λ_2. If $\lambda_1 > 0$, there are no equilibrium points. If $\lambda_1 < 0$, there are two equilibria

$$x^\star_\pm = (\pm\sqrt{-\lambda_1}, 0). \tag{2.4.15}$$

The linearization of f at $x^\star_+$ has the eigenvalues

$$\frac{\lambda_2 + \sqrt{-\lambda_1}}{2} \pm \sqrt{\frac{(\lambda_2 + \sqrt{-\lambda_1})^2}{4} + 2\sqrt{-\lambda_1}}, \tag{2.4.16}$$

and thus $x^\star_+$ is always a saddle. As for $x^\star_-$, the eigenvalues are

$$\frac{\lambda_2 - \sqrt{-\lambda_1}}{2} \pm \sqrt{\frac{(\lambda_2 - \sqrt{-\lambda_1})^2}{4} - 2\sqrt{-\lambda_1}}, \tag{2.4.17}$$

which shows that $x^\star_-$ is a node or a focus, unless $\lambda_2 = \sqrt{-\lambda_1}$. At $\lambda_2 = \sqrt{-\lambda_1}$, there is a Hopf bifurcation since the eigenvalues (2.4.17) are imaginary. One can check that the normal form is such that an unstable periodic orbit appears for $\lambda_2 < \sqrt{-\lambda_1}$.

With these informations, one can already draw some phase portraits. We know that there are no equilibrium points for $\lambda_1 > 0$ (Fig. 2.20–1). Crossing the λ_2-axis corresponds to a saddle–node bifurcation, producing a saddle and a source if $\lambda_2 > 0$ (Fig. 2.20–2), and a saddle and a sink if $\lambda_2 < 0$ (Fig. 2.20–4). The source expels an unstable periodic orbit when the curve $\lambda_2 = \sqrt{-\lambda_1}$ is crossed (Fig. 2.20–3).

There must be another bifurcation allowing to connect the vector fields slightly below $\lambda_2 = \sqrt{-\lambda_1}$, which admit a periodic orbit, and slightly to the left of $\lambda_1 = 0$, $\lambda_2 < 0$,

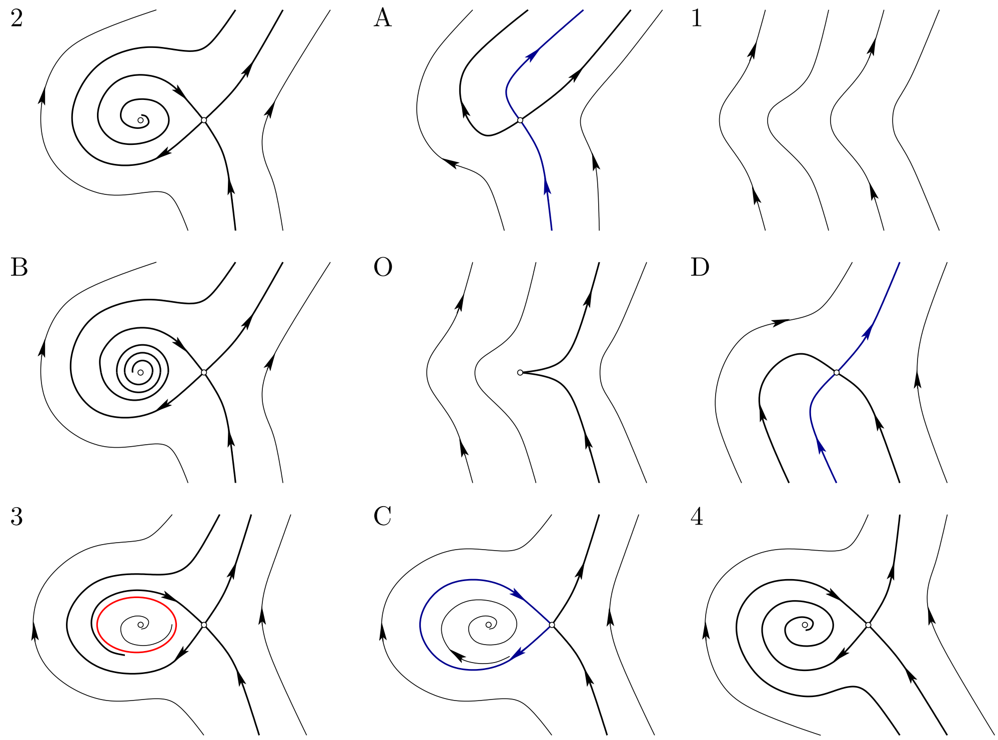

FIGURE 2.20. Unfolding the singular vector field (2.4.11) give rise to four structurally stable vector fields (1–4), and to four singular vector fields of codimension 1 (A–D).

which have no periodic orbit. The periodic orbit turns out to disappear in a homoclinic bifurcation, occurring on the curve

$$\lambda_2 = \frac{5}{7}\sqrt{-\lambda_1}\big[1 + \mathcal{O}(|\lambda_1|^{1/4})\big], \qquad \lambda_1 < 0, \tag{2.4.18}$$

see Fig. 2.19. We omit the proof of this fact, which is rather elaborate and relies on a rescaling, transforming the system into a close to Hamiltonian form, and on the computation of a Melnikov function, allowing to determine the splitting of the stable and unstable manifolds of the saddle.

It is remarkable that the double-zero eigenvalue, which is of purely local nature, gives rise to a global bifurcation. Bogdanov has proved that there are no other bifurcations in a neighbourhood of $(\lambda_1, \lambda_2) = 0$, and that the periodic orbit is unique. Thus any small perturbation g of the singular vector field (2.4.11) can be of one of the following structurally stable types (Fig. 2.20):

1. g has no equilibria or periodic orbits in a neighbourhood of the origin;
2. g admits a saddle and a source;
3. g admits a saddle, a sink and an unstable hyperbolic periodic orbit;
4. g admits a saddle and a sink;

or of one of the following codimension 1 singular types:

A. g has a ("semi-unstable") elementary saddle–node;
B. g has a saddle and an elementary composed focus;
C. g has a saddle admitting an elementary homoclinic loop;
D. g has a ("semi-stable") elementary saddle–node.

Observe how the codimension 2 singularity organizes several manifolds of codimension 1 singularities in its vicinity.

Other singular vector fields have been studied. For instance, the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1^2 x_2 \pm x_1^3\end{aligned} \tag{2.4.19}$$

is not a singularity of codimension 2 (it is conjectured to have codimension 4). However, Takens has shown that this vector field has codimension 2 *within* the class of vector fields which are invariant under the transformation $(x_1, x_2) \mapsto (-x_1, -x_2)$. It admits the unfolding

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \lambda_1 + \lambda_2 x_1 - x_1^2 x_2 \pm x_1^3\end{aligned} \tag{2.4.20}$$

Exercise 2.4.2. Find all equilibria of (2.4.20), determine their stability and the bifurcation curves. Try to sketch the phase portraits and to locate global bifurcations (the cases $+$ and $-$ are different).

2.5 Remarks on Dimension 3

We have just seen that the classification of two-dimensional vector fields is quite well developed. What is the situation in dimension 3?

In a sufficiently small neighbourhood of equilibrium points, the situation is very similar. Hyperbolic equilibrium points are *locally* insensitive to small perturbations. Codimension 1 bifurcations of equilibria are the same as in dimension 2, i.e., saddle–node and Hopf bifurcations. There is an additional candidate for a bifurcation of codimension 2, corresponding to a zero eigenvalue and a pair of imaginary eigenvalues, which is more difficult to analyze.

The dynamics near periodic orbits can be described by a two-dimensional Poincaré map. There are two new types of codimension 1 bifurcations: The period doubling bifurcation, which produces a new periodic orbit with twice the period of the old one, and a Hopf bifurcation, which can produce an invariant torus. Invariant tori constitute a new type of ω-limit set.

Up to this point, the complications due to the additional dimension seem manageable, and one could hope for a similar classification of three-dimensional vector fields as of two-dimensional ones. In particular, one has introduced a class of so-called *Morse–Smale systems*, which generalize to higher dimensions the vector fields satisfying Peixoto's theorem (in particular they have finitely many equilibria and periodic orbits, all hyperbolic). It was then conjectured that a vector field is structurally stable if and only if it is Morse–Smale, and that structurally stable vector fields are dense in the space of all vector fields.

These conjectures turned out to be false. In particular, Smale and others have constructed examples of vector fields that are structurally unstable, and such that any sufficiently small perturbation of them is still structurally unstable. Thus structurally *stable* vector fields are no longer dense in the space of all vector fields.

On the other hand, Morse–Smale systems are structurally stable but the converse is false. If Σ is a surface transverse to the flow, the dynamics of orbits returning to

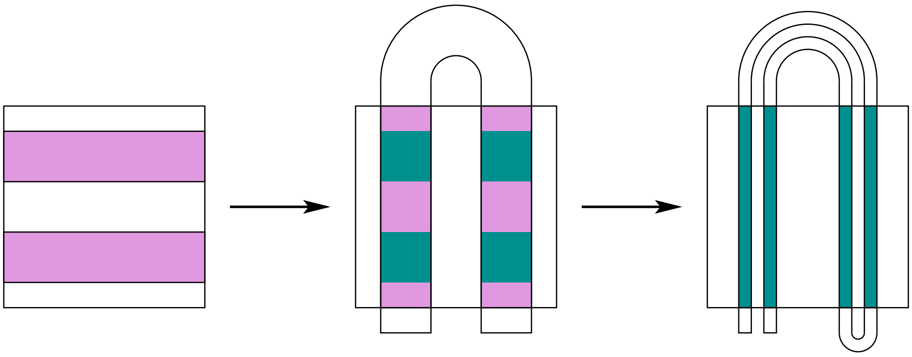

FIGURE 2.21. Smale's horseshoe map. Shaded horizontal rectangles are mapped to vertical ones. There is a countable infinity of periodic orbits, and an uncountable infinity of non-periodic orbits. Arbitrarily close points will become separated after a sufficient number of iterations.

Σ is described by a two-dimensional map. Smale has shown that if this map behaves qualitatively like the *horseshoe map* of Fig. 2.21, then it admits an invariant Cantor set. This set contains a countable infinity of hyperbolic periodic orbits and an uncountable infinity of non-periodic orbits, and thus it is not a Morse–Smale system. Nevertheless, the vector field restricted to this invariant set is structurally stable.

The invariant set of the horseshoe map exhibits very complicated dynamics (which is sensitive to small changes in the initial conditions), but the set is not attracting. Nevertheless, it has a strong influence on the dynamics in a neighbourhood. Later it was shown that there also exist ω-limit sets with such a complicated structure, called *strange attractors*. These constitute a new type of ω-limit set, and are generally *not* structurally stable.

Codimension 1 bifurcations of equilibria and periodic orbits do not produce complicated invariant sets. One could thus wonder where strange attractors come from, since it should be possible to produce these complicated systems by continuous deformation of a

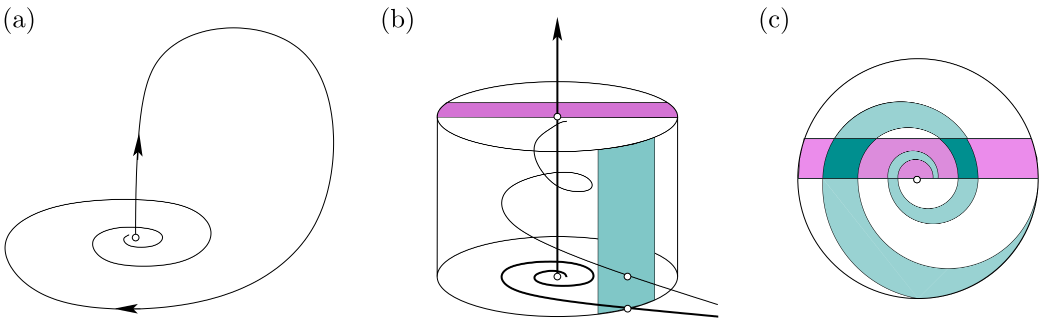

FIGURE 2.22. Šilnikov's example: (a) A saddle–focus with a homoclinic loop. (b) Consider a small cylinder whose bottom lies on the stable manifold of the saddle–focus. A rectangle on the side of the cylinder is the image of some strip on the top of the cylinder (the dimensions of the strip depend on the choice of radius and height of the cylinder). (c) The image of the rectangle is a spiral, which can be shown, under suitable hypotheses on the eigenvalues, to intersect the strip on the top of the cylinder in a horseshoe-like way. Thus the Poincaré map with respect to the top of the cylinder contains a horseshoe.

simple system. It turns out that horseshoes can appear in bifurcations involving a saddle connection. Such an example was first given by Šilnikov: Consider a saddle–focus, i.e., an equilibrium whose linearization admits one real eigenvalue α and two complex conjugate eigenvalues $\beta \pm \mathrm{i}\,\gamma$ (where α and β have opposite sign). If this equilibrium admits a saddle connection, Šilnikov has shown that under the condition $|\beta| < |\alpha|$, a perturbation of the flow admits a countable set of horseshoes (Fig. 2.22).

The theory developed in the two-dimensional case is still useful in the vicinity of equilibria and periodic orbits, but the global behaviour of three-dimensional flows may be substantially more difficult to analyze. We will encounter some specific examples in the next chapter.

Bibliographical comments

Most of the material discussed here, including the properties of equilibrium points and periodic orbits, limit sets, local and global bifurcations are treated in standard monographs such as [GH83] and [Wi90]. The classification of codimension 1 bifurcations is discussed more specifically in [HK91], though with only few proofs.

The proof of Peixoto's theorem is quite nicely exposed in [PP59]. Further developments can be found in [Pe62] (two-dimensional compact manifolds) and [Pe73] (classification of flows by graphs). A more advanced description of results related to structural stability and genericity is found in [PM82].

The Bogdanov–Takens bifurcation is originally described in [Ta74] and [Bo75]. The theory of unfoldings (or versal deformations) and relations to singularity theory are discussed in [Ar83]. There exist many partial results on local bifurcations of higher codimension, see for instance [KKR98].

The fact that structurally stable vector fields are not dense was first shown in [Sm66] and [PP68]. The example of Šilnikov originally appeared in [Si65], and is also discussed in [GH83] and [Wi90].

Chapter 3

Regular Perturbation Theory

In this chapter, we will consider non–autonomous ordinary differential equations of the form

$$\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \qquad x \in \mathcal{D} \subset \mathbb{R}^n, \tag{3.0.1}$$

where f and g are of class $\mathcal{C}^r$, $r \geqslant 1$, g is T-periodic in t, and ε is a small parameter. We will be mainly interested in the following question: If $x^\star$ is a stable equilibrium point of the unperturbed autonomous system $\dot{x} = f(x)$, does the perturbed system (3.0.1) admit a stable T-periodic orbit in a neighbourhood of $x^\star$? We will see below that this question frequently arises in applications.

Let us first recall the definition of stability:

Definition 3.0.1. *A periodic solution $\gamma(t)$ of* (3.0.1) *is called* stable *if for any $\delta > 0$, there exists $\eta > 0$ such that, if $\|x(0) - \gamma(0)\| \leqslant \eta$, then $\|x(t) - \gamma(t)\| \leqslant \delta$ for all $t \in \mathbb{R}$.*

It is easy to show that (3.0.1) admits a stable periodic solution if $x^\star$ is a sink:

Proposition 3.0.2. *Assume that all eigenvalues of $A = \frac{\partial f}{\partial x}(x^\star)$ have a strictly negative real part. Then there exist constants $\varepsilon_0, c_0 > 0$ such that, whenever $|\varepsilon| \leqslant \varepsilon_0$,* (3.0.1) *admits a unique stable T-periodic solution $\gamma(t)$ such that $\|\gamma(t) - x^\star\| \leqslant c_0\varepsilon$ for all $t \in \mathbb{R}$.*

PROOF: Let φ_t^ε denote the flow of (3.0.1), i.e., the solutions are written as $x(t) = \varphi_t^\varepsilon(x(0))$. We are interested in the application φ_T^ε, which can be interpreted as the Poincaré map of (3.0.1) in the extended phase space $\{(x, t) \in \mathcal{D} \times \mathbb{R}/T\mathbb{Z}\}$, taking the plane $t = 0$ as a Poincaré section. Indeed, since the equation is T-periodic in t, we can identify the planes $t = 0$ and $t = T$.

If $\varepsilon = 0$, we simply have $\frac{\partial}{\partial x}\varphi_t^0(x^\star) = \mathrm{e}^{At}$. Thus, if we introduce the function $\Phi(x, \varepsilon) = \varphi_T^\varepsilon(x) - x$, we have the relations

$$\Phi(x^\star, 0) = 0, \qquad \frac{\partial \Phi}{\partial x}(x^\star, 0) = \mathrm{e}^{AT} - \mathbb{1}.$$

Since all eigenvalues of A have a negative real part, e^{AT} has all its eigenvalues inside the unit circle, which means in particular that $\frac{\partial}{\partial x}\Phi(x^\star, 0)$ has no eigenvalue equal to zero. Thus the implicit function theorem implies the existence, for ε sufficiently small, of a unique $\mathcal{C}^r$ function $x^\star(\varepsilon)$ such that $\varphi_T^\varepsilon(x^\star) = x^\star$. This fixed point of the Poincaré map corresponds to a periodic orbit of (3.0.1).

For sufficiently small ε, the eigenvalues of $\frac{\partial}{\partial x}\varphi_T^\varepsilon(x^\star(\varepsilon))$ still lie inside the unit circle, which means that φ_T^ε is a contraction in some neighbourhood of $x^\star(\varepsilon)$. This implies the uniqueness of the periodic orbit. It also shows that $\|x(t) - \gamma(t)\|$ is smaller than $\|x(0)-\gamma(0)\|$ for $t/T \in \mathbb{N}$. One can show that $\|x(t)-\gamma(t)\|$ is smaller than a constant times $\|x(0)-\gamma(0)\|$ for *all* t by a standard argument on differential inequalities, see Corollary 3.1.7 below. This implies the stability of γ. $\square$

A similar argument shows that if $x^\star$ is a hyperbolic equilibrium point of f, then (3.0.1) admits a periodic orbit for sufficiently small ε, which is stable if and only if $x^\star$ is stable.

As we saw in Example 1.2.1, one often has to do with cases where $x^\star$ is elliptic. If the linearization A of f at $x^\star$ has imaginary eigenvalues, but none of them is a multiple of $2\pi\,\mathrm{i}/T$, the implicit function theorem can still be applied to prove the existence of a unique periodic orbit near $x^\star$. Analysing the stability of this orbit, however, turns out to be considerably more difficult, and will be one of the main goals of this chapter. Along the way, we will develop some methods which are useful in other problems as well.

Before discussing these methods, we will recall some properties of Hamiltonian systems, which form an important source of equations of the form (3.0.1), and provided the original motivation for their study.

3.1 Preliminaries

3.1.1 Hamiltonian Systems

Let $H \in \mathcal{C}^{r+1}(\mathcal{D},\mathbb{R})$, $r \geqslant 1$, be defined in an open subset $\mathcal{D}$ of $\mathbb{R}^{2n}$ (or on a $2n$-dimensional differentiable manifold). The integer n is called the *number of degrees of freedom*, and H is the *Hamilton function* or *Hamiltonian*. We denote elements of $\mathcal{D}$ by $(q,p) = (q_1,\dots,q_n,p_1,\dots,p_n)$. The *canonical equations* associated with H are defined as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad i = 1,\dots,n, \tag{3.1.1}$$

and define a *Hamiltonian flow* on $\mathcal{D}$. The simplest example of a Hamiltonian is

$$H = \frac{1}{2}\|p\|^2 + V(q), \tag{3.1.2}$$

whose canonical equations

$$\dot{q}_i = p_i, \qquad \dot{p}_i = -\frac{\partial V}{\partial q_i}(q) \tag{3.1.3}$$

are those of a particle in a potential V. The advantage of the canonical equations (3.1.1) is that they are invariant under a large class of coordinate transformations, and thus adapted to mechanical systems with constraints.

The "skew-symmetric" structure of the canonical equations (3.1.1), called *symplectic structure*, has some important consequences.

Proposition 3.1.1.

- *H is a constant of the motion.*
- *The Hamiltonian flow preserves the volume element* $\mathrm{d}q\,\mathrm{d}p = \mathrm{d}q_1\dots\mathrm{d}q_n\,\mathrm{d}p_1\dots\mathrm{d}p_n$.

- *A curve* $\gamma = \{q(t), p(t) \colon t_1 \leqslant t \leqslant t_2\}$ *corresponds to a solution of* (3.1.1) *if and only if the* action integral

$$\mathcal{S}(\gamma) = \int_\gamma p \cdot \mathrm{d}q - H \,\mathrm{d}t := \int_{t_1}^{t_2} \big[p(t) \cdot \dot{q}(t) - H(q(t), p(t))\big] \,\mathrm{d}t \tag{3.1.4}$$

is stationary with respect to variations of γ *(with fixed end points).*

PROOF:

1. By a direct calculation,

$$\frac{\mathrm{d}}{\mathrm{d}t} H(q(t), p(t)) = \sum_{i=1}^n \Big(\frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i\Big) = 0.$$

2. In the extended phase space $\mathcal{D} \times \mathbb{R}$, the canonical equations define the vector field $f(q, p, t) = (\partial H/\partial p, -\partial H/\partial q, 1)$. Let $M(0)$ be a compact set in $\mathcal{D}$, and $M(t)$ the image of $M(0)$ under the flow φ_t. Define a cylinder $\mathcal{C} = \{(M(s), s) \colon 0 \leqslant s \leqslant t\}$. Then we have, by Gauss' theorem,

$$\int_{M(t)} \mathrm{d}q \,\mathrm{d}p - \int_{M(0)} \mathrm{d}q \,\mathrm{d}p = \int_{\partial\mathcal{C}} f \cdot \mathrm{d}n = \int_{\mathcal{C}} (\nabla \cdot f) \,\mathrm{d}p \,\mathrm{d}q \,\mathrm{d}t = 0,$$

where $\mathrm{d}n$ is a normal vector to $\partial\mathcal{C}$, and we have used the fact that $f \cdot \mathrm{d}n = 0$ on the sides of the cylinder.

3. Let $\delta = \{\xi(t), \eta(t) \colon t_1 \leqslant t \leqslant t_2\}$ be a curve vanishing for $t = t_1$ and $t = t_2$. The first variation of the action in the direction δ is defined as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \mathcal{S}(\gamma + \varepsilon\delta)\Big|_{\varepsilon=0} &= \int_{t_1}^{t_2} \Big(p\dot{\xi} + \dot{q}\eta - \frac{\partial H}{\partial q}\xi - \frac{\partial H}{\partial p}\eta\Big) \,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \xi\Big(-\dot{p} - \frac{\partial H}{\partial q}\Big) + \eta\Big(\dot{q} - \frac{\partial H}{\partial p}\Big) \,\mathrm{d}t,\end{aligned}$$

where we have integrated the term $p\dot{\xi}$ by parts. Now it is clear that this expression vanishes for all perturbations δ if and only if $(q(t), p(t))$ satisfies the canonical equations (3.1.1). □

A useful notation is the *Poisson bracket* of two differentiable functions $f, g : \mathcal{D} \to \mathbb{R}$, defined as

$$\{f; g\} = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}. \tag{3.1.5}$$

Then we have, in particular,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \sum_{i=1}^n \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i = \{f; H\}. \tag{3.1.6}$$

Since $\{;\}$ is an antisymmetric bilinear form, we immediately recover that $\dot{H} = \{H; H\} = 0$. A function $J : \mathcal{D} \to \mathbb{R}$ is a *constant of motion* if and only if $\{J; H\} = 0$. One can show that if J_1 and J_2 are two constants of motion, then $\{J_1; J_2\}$ is also a constant of motion. J_1 and J_2 are *in involution* if $\{J_1; J_2\} = 0$.

For one degree of freedom ($n = 1$), orbits of the Hamiltonian flow are simply level curves of H. With more degrees of freedom, the manifolds $H = constant$ are invariant, but the motion within each manifold may be complicated. If there are additional constants of motion, however, there will be invariant manifolds of smaller dimension, in which the motion may be easier to analyse. To do this, we would like to choose coordinates which exploit these constants of motion, but keeping the symplectic structure of the differential equation. Transformations which achieve this are called *canonical transformations.*

Let $S(q, Q)$ be twice continuously differentiable in an open domain of $\mathbb{R}^{2n}$, such that

$$\det \frac{\partial^2 S}{\partial q \partial Q} \neq 0 \qquad \forall q, Q. \tag{3.1.7}$$

S is called a *generating function.* We introduce

$$\begin{aligned} p(q, Q) &= \frac{\partial S}{\partial q}(q, Q), \\ P(q, Q) &= -\frac{\partial S}{\partial Q}(q, Q). \end{aligned} \tag{3.1.8}$$

By the assumption (3.1.7), we can invert the relation $p = p(q, Q)$ with respect to Q, and thus the relations (3.1.8) define a transformation $(q, p) \mapsto (Q, P)$ in some open domain in $\mathbb{R}^{2n}$.

Proposition 3.1.2. *The transformation $(q, p) \mapsto (Q, P)$ is volume-preserving and canonical. The new equations of motion are given by*

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}, \qquad i = 1, \dots n, \tag{3.1.9}$$

where $K(Q, P) = H(q, p)$ is the old Hamiltonian expressed in the new variables.

PROOF: Note that $\mathrm{d}S = \sum_i p_i \,\mathrm{d}q_i - P_i \,\mathrm{d}Q_i$. For every i and a given compact $M \subset \mathbb{R}^2$, we have by Green's theorem

$$\int_M \mathrm{d}q_i \,\mathrm{d}p_i - \mathrm{d}Q_i \,\mathrm{d}P_i = \int_{\partial M} p_i \,\mathrm{d}q_i - P_i \,\mathrm{d}Q_i = \int_{\partial M} \frac{\partial S}{\partial q_i} \,\mathrm{d}q_i + \frac{\partial S}{\partial Q_i} \,\mathrm{d}Q_i = \int_{\partial M} \mathrm{d}S,$$

but this last integral vanishes since ∂M consists of one or several closed curves. This proves that $\mathrm{d}q_i \,\mathrm{d}p_i = \mathrm{d}Q_i \,\mathrm{d}P_i$ for every i. Furthermore, for any solution of the canonical equations, we have

$$\int_{t_1}^{t_2} P \cdot \mathrm{d}Q - H \,\mathrm{d}t = \int_{t_1}^{t_2} (p \cdot \mathrm{d}q - H \,\mathrm{d}t) - \int_{t_1}^{t_2} \mathrm{d}S.$$

The first integral on the right-hand side is stationary, and the second depends only on the end points. Thus the integral of $P \,\mathrm{d}Q - H \,\mathrm{d}t$ is stationary with respect to variations of the curve, and by Proposition 3.1.1, (Q, P) satisfies the canonical equations (3.1.9). □

Consider for the moment the case $n = 1$. As we have seen, the orbits are level curves of H. It would thus be useful to construct a canonical transformation such that one of the new variables is constant on each invariant curve. Such variables are called *action–angle variables.*

Assume that the level curve $H^{-1}(h) = \{(q,p) \colon H(q,p) = h\}$ is bounded. Then its *action* is defined by

$$I(h) = \frac{1}{2\pi} \int_{H^{-1}(h)} p \, \mathrm{d}q. \tag{3.1.10}$$

Assume further that $I'(h) \neq 0$ on some interval, so that the map $I = I(h)$ can be inverted to give $h = h(I)$. If $p(h,q)$ is the solution of $H(q,p) = h$ with respect to p, we can define a generating function

$$S(I,q) = \int_{q_0}^{q} p(h(I), q') \, \mathrm{d}q'. \tag{3.1.11}$$

(The function $p(h,q)$ is not uniquely defined in general, but the definition (3.1.11) makes sense for any parametrization of the curve $H(q,p) = h$). The transformation from (q,p) to action–angle variables (I,φ) is defined implicitly by

$$p(I,q) = \frac{\partial S}{\partial q}, \qquad \varphi(I,q) = \frac{\partial S}{\partial I}. \tag{3.1.12}$$

Note that the normalization in (3.1.10) has been chosen in such a way that $\varphi \to 2\pi-$ as $q \to q_0-$, and thus φ is indeed an angle. The Hamiltonian in action–angle variables takes the form

$$K(\varphi, I) = h(I), \tag{3.1.13}$$

and the associated canonical equations are simply

$$\dot{\varphi} = \frac{\partial K}{\partial I} = h'(I), \qquad \dot{I} = -\frac{\partial K}{\partial \varphi} = 0. \tag{3.1.14}$$

The first relation means that the angle φ is an "equal-time" parametrization of the level curve. Indeed, we have

$$\varphi(t) = \int_{q_0}^{q(t)} \frac{\partial p}{\partial H}(h(I), q') \, \mathrm{d}q' \frac{\mathrm{d}h}{\mathrm{d}I} = \int_{q_0}^{q(t)} \frac{\mathrm{d}q'}{\dot{q}} \frac{\mathrm{d}h}{\mathrm{d}I} = (t - t_0) \frac{\mathrm{d}h}{\mathrm{d}I}. \tag{3.1.15}$$

In the case $n > 1$, there is no reason for there to be other constants of motion than H (and functions of H). There are, however, a certain number of cases where additional constants of motion exist, for instance when H is invariant under a (continuous) symmetry group (Noether's theorem). A Hamiltonian with n degrees of freedom is called *integrable* if it has n constants of motion $J_1, \dots, J_n$ such that

- the J_i are in involution, i.e., $\{J_i; J_k\} = 0$ for all (i,k);
- the gradients of the J_i are everywhere linearly independent in $\mathbb{R}^{2n}$.

Arnol'd has shown that if a given level set of $J_1, \dots, J_n$ is compact, then it typically has the topology of an n-dimensional torus. One can then introduce action–angle variables in a similar way as for $n = 1$, and the Hamiltonian takes the form

$$K(\varphi, I) = H_0(I). \tag{3.1.16}$$

Consider now a perturbation of the integrable Hamiltonian, of the form

$$H(\varphi, I, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon). \tag{3.1.17}$$

The associated canonical equations are

$$\dot{\varphi}_j = \frac{\partial H_0}{\partial I_j} + \varepsilon \frac{\partial H_1}{\partial I_j}, \qquad \dot{I}_j = -\varepsilon \frac{\partial H_1}{\partial \varphi_j}. \tag{3.1.18}$$

We can now make the link with the system $\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon)$ introduced at the beginning of this chapter. A first possibility is to take $x = (\varphi, I)$, $f = (\partial H_0/\partial I, 0)$ and $g = (\partial H_1/\partial I, -\partial H_1/\partial \varphi)$. Then g is not time-dependent. There are, however, other possibilities.

Example 3.1.3. Assume for instance that $\partial H_0/\partial I_1 \neq 0$ for all I. If $\partial H_1/\partial I_1$ is bounded, then φ_1 will be a monotonous function of t for sufficiently small ε. We can thus replace t by φ_1 and consider the $2n-1$ equations

$$\begin{aligned} \frac{\mathrm{d}\varphi_k}{\mathrm{d}\varphi_1} &= \frac{\partial H_0/\partial I_k + \varepsilon \partial H_1/\partial I_k}{\partial H_0/\partial I_1 + \varepsilon \partial H_1/\partial I_1}, & k &= 2, \dots, n, \\ \frac{\mathrm{d}I_j}{\mathrm{d}\varphi_1} &= -\varepsilon \frac{\partial H_1/\partial \varphi_j}{\partial H_0/\partial I_1 + \varepsilon \partial H_1/\partial I_1}, & j &= 1, \dots, n. \end{aligned} \tag{3.1.19}$$

We thus obtain a system of the form $\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon)$ for $x = (\varphi_2, \dots, \varphi_n, I_1, \dots, I_n)$, and with $f = ((\partial H_0/\partial I)/(\partial H_0/\partial I_1), 0)$. The remainder g depends periodically on φ_1 because φ_1 is an angle.

Example 3.1.4. Assume that $\gamma(t) = (q(t), p(t))$ is a T-periodic solution of a Hamiltonian system (which need not be close to integrable). The variable $y = x - \gamma(t)$ satisfies an equation of the form

$$\dot{y} = A(t)y + g(y, t), \tag{3.1.20}$$

where

$$A(t) = \begin{pmatrix} \partial^2 H/\partial q \partial p & \partial^2 H/\partial p^2 \\ -\partial^2 H/\partial q^2 & -\partial^2 H/\partial q \partial p \end{pmatrix}\Bigg|_{\gamma(t)} \tag{3.1.21}$$

and there are constants $M, \delta > 0$ such that $\|g(y, t)\| \leqslant M\|y\|^2$ for $\|y\| \leqslant \delta$. The equation (3.1.20) is not yet in the form we are looking for, but we are going to transform it now. Let $U(t)$ be the solution of

$$\dot{U} = A(t)U, \qquad U(0) = \mathbb{1}. \tag{3.1.22}$$

Since A is T-periodic, we know by Floquet's theorem that $U(t) = P(t)\,\mathrm{e}^{tB}$ for some T-periodic matrix $P(t)$. By substitution into (3.1.22), we see that

$$\dot{P} = A(t)P - PB, \tag{3.1.23}$$

and thus the change of variables $y = P(t)z$ yields the equation

$$\dot{z} = Bz + P^{-1}g(Pz, t). \tag{3.1.24}$$

Finally we carry out a rescaling $z = \varepsilon w$, which means that we zoom on a small neighbourhood of the periodic orbit. Here ε is a small positive parameter, and the change of variables is defined for all $\varepsilon > 0$. w obeys the equation

$$\dot{w} = Bw + \varepsilon G(w, t, \varepsilon), \tag{3.1.25}$$

which is exactly of the form (3.0.1), with

$$\begin{aligned} G(w,t,\varepsilon) &= \frac{1}{\varepsilon^2}P^{-1}g(\varepsilon Pw,t), \\ \|G(w,t,\varepsilon)\| &\leqslant \frac{1}{\varepsilon^2}\|P^{-1}\|M\|\varepsilon Pw\|^2 \leqslant M\|P^{-1}\|\|P\|\|w\|^2. \end{aligned} \tag{3.1.26}$$

In fact, (3.1.20) and (3.1.24) can both be considered as small perturbations of a linear system, deriving from a (possibly time-dependent) quadratic Hamiltonian. Since a quadratic form can always be diagonalized, these linear equations are integrable, and thus the motion in a small neighbourhood of a periodic orbit can be viewed as a perturbed integrable system.

3.1.2 Basic Estimates

Let us return to the equation

$$\dot{x} = f(x) + \varepsilon g(x,t,\varepsilon), \qquad x \in \mathcal{D} \subset \mathbb{R}^n, \tag{3.1.27}$$

where f and g are of class $\mathcal{C}^r$ for some $r \geqslant 1$. Our aim is to describe the difference between the dynamics for $\varepsilon = 0$ and $\varepsilon > 0$. Let us recall a fundamental theorem on ordinary differential equations, and the dependence of their solutions on parameters.

Theorem 3.1.5. *Let $F(x,t,\lambda) \in \mathcal{C}^r(\mathcal{D}_0,\mathbb{R}^n)$, $r \geqslant 1$, for an open set $\mathcal{D}_0 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$. Then the solution of $\dot{x} = F(x,t,\lambda)$ with initial condition $x(t_0) = x_0$ is a $\mathcal{C}^r$ function of x_0, t_0, t and λ on its domain of existence.*

It follows that $x(t)$ admits an expansion of the form

$$x(t) = x^0(t) + \varepsilon x^1(t) + \varepsilon^2 x^2(t) + \dots + \varepsilon^r x^r(t,\varepsilon) \tag{3.1.28}$$

in some intervals of t and ε. In particular, $x^0(t)$ is a solution of the unperturbed equation $\dot{x} = f(x)$, and thus the solution of (3.1.27) stays close to $x^0(t)$ on compact time intervals.

A useful tool to estimate the time-dependence of the error $x(t) - x^0(t)$ is the following lemma from the theory of differential inequalities.

Lemma 3.1.6 (Gronwall's inequality). *Let φ, α and β be continuous real-valued functions on $[a,b]$, with β non-negative, and assume that*

$$\varphi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\varphi(s)\,\mathrm{d}s \qquad \forall t \in [a,b]. \tag{3.1.29}$$

Then

$$\varphi(t) \leqslant \alpha(t) + \int_a^t \beta(s)\alpha(s)\,\mathrm{e}^{\int_s^t \beta(u)\,\mathrm{d}u}\,\mathrm{d}s \qquad \forall t \in [a,b]. \tag{3.1.30}$$

PROOF: Let

$$R(t) = \int_a^t \beta(s)\varphi(s)\,\mathrm{d}s.$$

Then $\varphi(t) \leqslant \alpha(t) + R(t)$ for all $t \in [a,b]$ and, since $\beta(t) \geqslant 0$,

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = \beta(t)\varphi(t) \leqslant \beta(t)\alpha(t) + \beta(t)R(t).$$

Let $B(s) = \int_a^s \beta(u)\,\mathrm{d}u$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{e}^{-B(s)}\,R(s) = \bigl[R'(s) - \beta(s)R(s)\bigr]\,\mathrm{e}^{-B(s)} \leqslant \beta(s)\alpha(s)\,\mathrm{e}^{-B(s)}$$

and thus, integrating from a to t,

$$\mathrm{e}^{-B(t)}\,R(t) \leqslant \int_a^t \beta(s)\alpha(s)\,\mathrm{e}^{-B(s)}\,\mathrm{d}s.$$

We obtain the conclusion by multiplying this expression by $\mathrm{e}^{B(t)}$ and inserting the result into (3.1.29). □

Corollary 3.1.7. *Assume f admits a uniform Lipschitz constant K in $\mathcal{D}$, and $\|g\|$ is uniformly bounded by M in $\mathcal{D} \times \mathbb{R} \times [0, \varepsilon_0]$. If $x(0) = x^0(0)$, then*

$$\|x(t) - x^0(t)\| \leqslant \frac{\varepsilon M}{K}\bigl[\mathrm{e}^{Kt} - 1\bigr] \qquad \forall t \geqslant 0. \tag{3.1.31}$$

Proof: Let $y = x - x^0(t)$. Then

$$\dot{y} = f(x^0(t) + y) - f(x^0(t)) + \varepsilon g(x(t), t, \varepsilon).$$

Integrating between 0 and t, we obtain

$$y(t) = \int_0^t \bigl[f(x^0(s) + y(s)) - f(x^0(s))\bigr]\,\mathrm{d}s + \varepsilon \int_0^t g(x(s), s, \varepsilon)\,\mathrm{d}s.$$

Taking the norm, we arrive at the estimate

$$\begin{aligned}
\|y(t)\| &\leqslant \int_0^t \|f(x^0(s) + y(s)) - f(x^0(s))\|\,\mathrm{d}s + \varepsilon \int_0^t \|g(x(s), s, \varepsilon)\|\,\mathrm{d}s \\
&\leqslant \int_0^t K\|y(s)\|\,\mathrm{d}s + \varepsilon M t.
\end{aligned}$$

Applying Gronwall's inequality, we obtain (3.1.31), as desired. □

The estimate (3.1.31) shows that for a given t, we can make the deviation $\|x(t) - x^0(t)\|$ small by taking ε small. However, since e^{Kt} grows without bound as $t \to \infty$, we cannot make any statements on stability. Of course, (3.1.31) is only an upper bound. The following example shows, however, that it cannot be improved without further assumptions on f.

Example 3.1.8. Consider, as a special case of (3.1.25), the equation

$$\dot{x} = Bx + \varepsilon g(t), \tag{3.1.32}$$

where g is T-periodic in t. Let us assume that B is in diagonal form, with (possibly complex) eigenvalues $b_1, \dots, b_n$. Then the solution can be written, for each component, as

$$x_j(t) = \mathrm{e}^{b_j t}\,x_j(0) + \varepsilon \int_0^t \mathrm{e}^{b_j(t-s)}\,g_j(s)\,\mathrm{d}s. \tag{3.1.33}$$

If the g_j are written in Fourier series as

$$g_j(s) = \sum_{k\in\mathbb{Z}} c_{jk}\,\mathrm{e}^{\mathrm{i}\,\omega k s}, \qquad\qquad \omega = \frac{2\pi}{T}, \tag{3.1.34}$$

we have to compute integrals of the form

$$\int_0^t \mathrm{e}^{b_j(t-s)}\,\mathrm{e}^{\mathrm{i}\,\omega k s}\,\mathrm{d}s = \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}\,\omega k t} - \mathrm{e}^{b_j t}}{\mathrm{i}\,\omega k - b_j} & \text{if } b_j \neq \mathrm{i}\,\omega k, \\ t\,\mathrm{e}^{b_j t} & \text{if } b_j = \mathrm{i}\,\omega k. \end{cases} \tag{3.1.35}$$

If $\operatorname{Re} b_j \neq 0$, solutions grow or decrease exponentially fast, just like for $\varepsilon = 0$. If $\operatorname{Re} b_j = 0$, however, the origin is a stable equilibrium point of the unperturbed system, while the perturbed system displays the phenomenon of *resonance*: If b_j is a multiple of $\mathrm{i}\,\omega$, the deviation grows linearly with t; if not, it remains bounded, but the amplitude may become large if the denominator $\mathrm{i}\,\omega k - b_j$ is small for some k.

3.2 Averaging and Iterative Methods

Corollary 3.1.7 gives a rather rough estimate on the difference between solutions of the perturbed and unperturbed equations. There exist a number of iterative methods that allow to improve these estimates, by replacing the unperturbed equation by a better approximation of the perturbed one, which still remains easier to solve.

3.2.1 Averaging

The method of averaging allows to deal with differential equations depending periodically on time, by comparing their solutions with a simpler, autonomous equation. The method applies to systems of the form

$$\dot{x} = \varepsilon g(x, t, \varepsilon), \tag{3.2.1}$$

where g is T-periodic in t.

We are, in fact, interested in more general equations of the form

$$\dot{x} = f(x) + \varepsilon g(x, t, \varepsilon), \tag{3.2.2}$$

but it is often possible to transform this equation into (3.2.1). Indeed, let $h(x, t, \varepsilon) \in \mathcal{C}^r(\mathcal{D} \times \mathbb{R}/T\mathbb{Z} \times [0, \varepsilon_0], \mathbb{R}^n)$ be T-periodic in t, and invertible in x near $x = 0$. Then the variable $y = h(x, t, \varepsilon)$ satisfies

$$\dot{y} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(x) + \varepsilon \frac{\partial h}{\partial x} g(x, t, \varepsilon). \tag{3.2.3}$$

Thus, if we manage to find an h such that

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(x) = \mathcal{O}(\varepsilon), \tag{3.2.4}$$

then y satisfies an equation of the form (3.2.1). This is, in fact, possible if the unperturbed equation has periodic solutions with a period close to T, i.e., near resonance.

Example 3.2.1. Consider the equation

$$\dot{x} = B(\omega_0)x + \varepsilon g(x,t,\varepsilon), \qquad B(\omega_0) = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{pmatrix}. \tag{3.2.5}$$

The solutions of the unperturbed equation are of the form

$$x^0(t) = \mathrm{e}^{B(\omega_0)t}\, x^0(0), \qquad \mathrm{e}^{B(\omega_0)t} = \begin{pmatrix} \cos(\omega_0 t) & \frac{1}{\omega_0}\sin(\omega_0 t) \\ -\omega_0\sin(\omega_0 t) & \cos(\omega_0 t) \end{pmatrix}, \tag{3.2.6}$$

and are thus periodic with period $2\pi/\omega_0$. Now let $\omega = 2\pi/T$ and take $h(x,t) = \mathrm{e}^{-B(\omega)t}\, x$. We obtain that

$$\frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} f(x) = \mathrm{e}^{-B(\omega)t}\bigl[B(\omega_0) - B(\omega)\bigr]x. \tag{3.2.7}$$

Thus if ω_0 is close to ω, that is, if $\omega^2 = \omega_0^2 + \mathcal{O}(\varepsilon)$, then (3.2.7) is indeed of order ε. More generally, taking $h(x,t) = \mathrm{e}^{-B(\omega/k)t}\, x$ for some $k \in \mathbb{Z}$ allows to treat cases where $\omega^2 = k^2\omega_0^2 + \mathcal{O}(\varepsilon)$. This trick is known as the *van der Pol transformation*.

Let us now return to the equation

$$\dot{x} = \varepsilon g(x,t,\varepsilon). \tag{3.2.8}$$

We assume that $g \in \mathcal{C}^r(\mathcal{D} \times \mathbb{R}/T\mathbb{Z} \times [0,\varepsilon_0], \mathbb{R}^n)$, where $\mathcal{D}$ is a compact subset of $\mathbb{R}^n$ and $r \geqslant 2$. Note that if M is an upper bound for $\|g\|$ in $\mathcal{D}$, then $\|x(t) - x(0)\| \leqslant \varepsilon M t$. Thus the trivial approximation $\dot{x} = 0$ of (3.2.8) is good to order ε for times of order 1 (this is, in fact, a particular case of Corollary 3.1.7 in the limit $K \to 0$). A better approximation is given by the averaged equation:

Definition 3.2.2. *The* averaged system *associated with* (3.2.8) *is the autonomous equation*

$$\dot{y}^0 = \varepsilon\langle g\rangle(y^0), \qquad \langle g\rangle(x) = \frac{1}{T}\int_0^T g(x,t,0)\,\mathrm{d}t. \tag{3.2.9}$$

The averaging theorem shows that (3.2.9) is a good approximation of (3.2.8) up to times of order $1/\varepsilon$:

Theorem 3.2.3. *There exists a $\mathcal{C}^r$ change of coordinates $x = y + \varepsilon w(y,t)$, with w a T-periodic function of t, transforming* (3.2.8) *into*

$$\dot{y} = \varepsilon\langle g\rangle(y) + \varepsilon^2 g_1(y,t,\varepsilon), \tag{3.2.10}$$

with g_1 also T-periodic in t. Moreover,

- *if $x(t)$ and $y^0(t)$ are solutions of the original and averaged systems* (3.2.8) *and* (3.2.9) *respectively, with initial conditions $x(0) = y^0(0) + \mathcal{O}(\varepsilon)$, then $x(t) = y^0(t) + \mathcal{O}(\varepsilon)$ for times t of order $1/\varepsilon$;*
- *if $y^\star$ is an equilibrium point of* (3.2.9) *such that $\frac{\partial}{\partial y}\langle g\rangle(y^\star)$ has no eigenvalue equal to zero, then* (3.2.8) *admits a T-periodic solution $\gamma_\varepsilon(t) = y^\star + \mathcal{O}(\varepsilon)$ for sufficiently small ε; if, in addition, $\frac{\partial}{\partial y}\langle g\rangle(y^\star)$ has no imaginary eigenvalue, then $\gamma_\varepsilon(t)$ has the same type of stability as $y^\star$ for sufficiently small ε.*

Proof:

- We begin by constructing the change of variables explicitly. We first split g into its averaged and oscillating part:
$$g(x,t,\varepsilon) = \langle g\rangle(x) + g^0(x,t,\varepsilon), \qquad\qquad \langle g^0\rangle(x) = 0.$$
Inserting $x = y + \varepsilon w(y,t)$ into (3.2.8), we get
$$\dot x = \Bigl[\mathbb{1} + \varepsilon\frac{\partial w}{\partial y}\Bigr]\dot y + \varepsilon\frac{\partial w}{\partial t}(y,t) = \varepsilon\langle g\rangle(y+\varepsilon w) + \varepsilon g^0(y+\varepsilon w,t,\varepsilon),$$
and expanding $\dot y$ into Taylor series, we obtain
$$\dot y = \varepsilon\bigl[\langle g\rangle(y) + g^0(y,t,0) - \frac{\partial w}{\partial t}(y,t)\bigr] + \mathcal{O}(\varepsilon^2),$$
where the remainder of order ε^2 is a T-periodic function of t, the Taylor expansion of which can be computed if necessary. Now choosing
$$w(y,t) = \int_0^t g^0(y,s,0)\,\mathrm{d}s$$
yields the desired form for $\dot y$. Note that w is indeed T-periodic in t since g^0 has average zero.
- The difference $z(t) = y(t) - y^0(t)$ satisfies the equation
$$\dot z = \varepsilon\bigl[\langle g\rangle(y) - \langle g\rangle(y^0)\bigr] + \varepsilon^2 g_1(y,t,\varepsilon),$$
and thus we can proceed as in the proof of Corollary 3.1.7. Since
$$z(t) = z(0) + \int_0^t \Bigl\{\varepsilon\bigl[\langle g\rangle(y(s)) - \langle g\rangle(y^0(s))\bigr] + \varepsilon^2 g_1(y(s),s,\varepsilon)\Bigr\}\,\mathrm{d}s,$$
taking K as a Lipschitz constant for $\langle g\rangle$ and M as a bound for $\|g_1\|$ (which must exist since $\mathcal{D}$ is compact), we arrive at the estimate
$$\|z(t)\| \leqslant \|z(0)\| + \int_0^t \varepsilon K\|z(s)\|\,\mathrm{d}s + \varepsilon^2 Mt,$$
and Gronwall's inequality provides us with the bound
$$\|z(t)\| \leqslant \|z(0)\|\,\mathrm{e}^{\varepsilon Kt} + \frac{\varepsilon M}{K}\bigl(\mathrm{e}^{\varepsilon Kt} - 1\bigr).$$
Since we have assumed that $\|z(0)\| = \mathcal{O}(\varepsilon)$, the error made by the averaging approximation remains of order ε for $t \leqslant 1/(K\varepsilon)$.
- The assertion on the periodic orbit is proved in a similar way as we did in Proposition 3.0.2. Let φ_t denote the flow of the transformed equation (3.2.10), and let φ_t^0 denote the flow of the averaged equation (3.2.9) (φ^0 is obtained from φ by letting $g_1 = 0$, *not* by letting $\varepsilon = 0$). By the fundamental Theorem 3.1.5, these flows may be Taylor-expanded in ε up to order ε^2. Setting $t = T$, we obtain the Poincaré maps of both systems
$$\begin{aligned}\varphi_T^0(y) &=: P^0(y,\varepsilon) = y + \varepsilon P_1^0(y) + \varepsilon^2 P_2^0(y,\varepsilon)\\ \varphi_T(y) &=: P(y,\varepsilon) = y + \varepsilon P_1(y) + \varepsilon^2 P_2(y,\varepsilon).\end{aligned}$$

By our estimate on $\|z(t)\|$, we must have $P_1(y) = P_1^0(y)$. If $y^\star$ is an equilibrium of (3.2.10) (that is, if $\langle g\rangle(y^\star) = 0$), it is a fixed point of P^0, and in particular

$$P_1^0(y^\star) = 0.$$

If we denote $\frac{\partial}{\partial y}\langle g\rangle(y^\star)$ by A, we also have

$$\frac{\partial P^0}{\partial y}(y^\star, \varepsilon) = \mathrm{e}^{\varepsilon AT} = \mathbb{1} + \varepsilon AT + \mathcal{O}(\varepsilon^2) \qquad \Rightarrow \qquad \frac{\partial P_1^0}{\partial y}(y^\star) = AT.$$

Now if we define the function

$$\Phi(y, \varepsilon) = P_1(y) + \varepsilon P_2(y, \varepsilon),$$

the above relations imply that

$$\Phi(y^\star, 0) = 0, \qquad\qquad \frac{\partial \Phi}{\partial y}(y^\star, 0) = AT.$$

Thus if A has no vanishing eigenvalue, the implicit function theorem implies the existence, for small ε, of a fixed point $y^\star(\varepsilon)$ of P. This fixed point corresponds to a periodic orbit γ_ε, which remains in an ε-neighbourhood of $y^\star$ because of our estimate on $\|z\|$. Since $x = y + \varepsilon w(y, t)$, the corresponding x-coordinates of the periodic orbit are also ε-close to $y^\star$.
Finally, if A has no imaginary eigenvalue, then the linearization of P^0 at $y^\star$ has no eigenvalue on the unit circle, and this remains true, for sufficiently small ε, for the linearization of P at $y^\star(\varepsilon)$. Thus the fixed points of P and P^0 have the same type of stability, and this property carries over to the periodic orbit. □

The averaged approximation of a periodic system is considerably easier to study, since it does not depend on time. In order to find periodic orbits of the initial system, it is sufficient to find equilibrium points of the averaged system. The approximation is reliable on a larger time scale than the unperturbed system. However, it does not give any information on stability if the averaged system has an elliptic equilibrium.

Example 3.2.4. Consider a small perturbation of a two-degree-of-freedom integrable system, whose Hamiltonian depends only on one action:

$$H(\varphi_1, \varphi_2, I_1, I_2, \varepsilon) = H_0(I_1) + \varepsilon H_1(\varphi_1, \varphi_2, I_1, I_2, \varepsilon). \tag{3.2.11}$$

This situation arises, for instance, in the Kepler problem of a planet orbiting the Sun. If $\varepsilon = 0$, φ_1 rotates with constant speed $\Omega(I_1) = H_0'(I_1)$ (as does the area swept by the planet in its orbit), while all other variables are fixed. If $\Omega(I_1) \neq 0$, we may write the equations of motion for sufficiently small ε as

$$\begin{aligned} \frac{\mathrm{d}\varphi_2}{\mathrm{d}\varphi_1} &= \varepsilon \frac{\partial H_1/\partial I_2}{\Omega(I_1) + \varepsilon \partial H_1/\partial I_1} \\ \frac{\mathrm{d}I_j}{\mathrm{d}\varphi_1} &= -\varepsilon \frac{\partial H_1/\partial \varphi_j}{\Omega(I_1) + \varepsilon \partial H_1/\partial I_1}, \qquad\qquad j = 1, 2. \end{aligned} \tag{3.2.12}$$

The averaged version of this system is

$$\begin{aligned}
\frac{\mathrm{d}\varphi_2}{\mathrm{d}\varphi_1} &= \frac{\varepsilon}{\Omega(I_1)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial H_1}{\partial I_2}(\varphi_1,\varphi_2,I_1,I_2,0)\,\mathrm{d}\varphi_1 \\
\frac{\mathrm{d}I_2}{\mathrm{d}\varphi_1} &= -\frac{\varepsilon}{\Omega(I_1)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial H_1}{\partial \varphi_2}(\varphi_1,\varphi_2,I_1,I_2,0)\,\mathrm{d}\varphi_1 \\
\frac{\mathrm{d}I_1}{\mathrm{d}\varphi_1} &= -\frac{\varepsilon}{\Omega(I_1)}\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial H_1}{\partial \varphi_1}(\varphi_1,\varphi_2,I_1,I_2,0)\,\mathrm{d}\varphi_1 = 0.
\end{aligned} \tag{3.2.13}$$

Now we can make an important observation. Consider the *averaged Hamiltonian*

$$\begin{aligned}
\langle H\rangle(\varphi_2,I_2;I_1,\varepsilon) &= \frac{1}{2\pi}\int_0^{2\pi} H(\varphi_1,\varphi_2,I_1,I_2,\varepsilon)\,\mathrm{d}\varphi_1 \\
&= H_0(I_1) + \varepsilon\langle H_1\rangle(\varphi_2,I_2;I_1,\varepsilon).
\end{aligned} \tag{3.2.14}$$

The ordering of the variables in (3.2.14) is meant to be suggestive, since we will obtain a closed system of equations for φ_2, I_2. Indeed, the associated canonical equations are

$$\begin{aligned}
\dot\varphi_1 &= \Omega(I_1) + \varepsilon\frac{\partial\langle H_1\rangle}{\partial I_1}(\varphi_2,I_2;I_1,\varepsilon) & \dot I_1 &= 0 \\
\dot\varphi_2 &= \varepsilon\frac{\partial\langle H_1\rangle}{\partial I_2}(\varphi_2,I_2;I_1,\varepsilon) & \dot I_2 &= -\varepsilon\frac{\partial\langle H_1\rangle}{\partial \varphi_2}(\varphi_2,I_2;I_1,\varepsilon).
\end{aligned} \tag{3.2.15}$$

At first order in ε, these equations are equivalent to (3.2.13). We conclude that for Hamiltonian systems of the form (3.2.11), averaging can be performed directly on the Hamiltonian, with respect to the “fast” variable φ_1.

Note that I_1 and $\langle H\rangle$ are constants of the motion for the averaged system. By the averaging theorem, they vary at most by an amount of order ε during time intervals of order $1/\varepsilon$. Such quantities are called *adiabatic invariants*.

Remark also that since the averaged Hamiltonian no longer depends on φ_1, the only dynamical variables are φ_2 and I_2, while I_1 plays the rôle of a parameter. The averaged Hamiltonian thus has one degree of freedom and is integrable. One can generalize the method to more degrees of freedom. In that case, the averaged system has one degree of freedom less than the original one. There also exist generalizations to systems with several fast variables.

The method of averaging of Hamiltonian systems with respect to fast variables is of major importance in applications. For instance, it allowed Laskar to integrate the motion of the Solar System over several hundred million years, while numerical integrations of the original system are only reliable on time spans of a few hundred to a few thousand years.

3.2.2 Iterative Methods

We have seen that the averaging procedure allows us to obtain a better approximation of the perturbed system, valid for longer times. A natural question is whether this method can be pushed further, in order to yield still better approximations. This is indeed possible to some extent. Consider for instance the equation

$$\dot y = \varepsilon\langle g\rangle(y) + \varepsilon^2 g_1(y,t,\varepsilon) \tag{3.2.16}$$

of the transformed system in Theorem 3.2.3. Assuming sufficient differentiability, and proceeding in the same way as when we constructed the change of variables $x = y+\varepsilon w(y,t)$, we can construct a new change of variables $y = y_2 + \varepsilon^2 w_2(y_2,t)$ such that

$$\dot{y}_2 = \varepsilon\langle g\rangle(y_2) + \varepsilon^2\langle g_1\rangle(y_2) + \varepsilon^3 g_2(y_2,t,\varepsilon). \tag{3.2.17}$$

The associated averaged system is

$$\dot{y}_2^0 = \varepsilon\langle g\rangle(y_2^0) + \varepsilon^2\langle g_1\rangle(y_2^0), \tag{3.2.18}$$

and if K_2 and M_2 denote, respectively, a Lipschitz constant for $\langle g\rangle + \varepsilon\langle g_1\rangle$ and an upper bound on $\|g_2\|$, one easily obtains from Gronwall's inequality that

$$\|y_2(t) - y_2^0(t)\| \leqslant \Bigl(\|y_2(0) - y_2^0(0)\| + \frac{\varepsilon^2 M_2}{K_2}\Bigr)\,\mathrm{e}^{\varepsilon K_2 t}. \tag{3.2.19}$$

Thus for an appropriate initial condition, the error remains of order ε^2 for times smaller than $1/(K_2\varepsilon)$. This procedure can be repeated as long as the system is differentiable enough, and after $r-1$ steps we get exact and approximated systems of the form

$$\begin{aligned} \dot{y}_r &= \varepsilon\langle g\rangle(y_r) + \dots + \varepsilon^r\langle g_{r-1}\rangle(y_r) + \varepsilon^{r+1} g_r(y_r,t,\varepsilon) \\ \dot{y}_r^0 &= \varepsilon\langle g\rangle(y_r^0) + \dots + \varepsilon^r\langle g_{r-1}\rangle(y_r^0), \end{aligned} \tag{3.2.20}$$

satisfying

$$\|y_r(t) - y_r^0(t)\| \leqslant \Bigl(\|y_r(0) - y_r^0(0)\| + \frac{\varepsilon^r M_r}{K_r}\Bigr)\,\mathrm{e}^{\varepsilon K_r t}. \tag{3.2.21}$$

For sufficiently small $\|y_r(0) - y_r^0(0)\|$, this error remains of order ε^r for times of order $1/(K_r\varepsilon)$. One could thus wonder, if the initial system is analytic, whether this procedure can be repeated infinitely often, in such a way that the time-dependence is eliminated completely. It turns out that this is *not* possible in general. The reason is that the bounds M_r and K_r can grow rapidly with r (typically, like $r!$), so that the amplitude of the error (3.2.21) becomes large, while the time interval on which the approximation is valid shrinks to zero.

Once again, we are forced to conclude that the method of averaging provides very useful approximations on a given bounded time scale, up to some power in ε, but fails in general to describe the long-term dynamics, especially in the neighbourhood of elliptic periodic orbits. We will return to this point in the next section.

Similar properties hold for Hamiltonian systems, for which these iterative methods were first developed. The main difference is that certain components of the averaged equations may vanish (see for instance (3.2.15) in Example 3.2.4), which implies that the averaged approximation may be valid for longer time intervals. Consider a Hamiltonian system of the form

$$H(I,\varphi,\varepsilon) = H_0(I) + \varepsilon H_1(I,\varphi) + \varepsilon^2 H_2(I,\varphi) + \cdots \tag{3.2.22}$$

appearing for instance when describing the dynamics of the Solar System. A method to simplify this system, which is due to von Zeipel, proceeds as follows:

- Let $W(\varphi,J,\varepsilon) = \varphi\cdot J + \varepsilon W_1(\varphi,J)$ be a generating function. It defines a near-identity canonical transformation $(\varphi,I) \mapsto (\psi,J)$ implicitly by

$$I = \frac{\partial W}{\partial\varphi} = J + \varepsilon\frac{\partial W_1}{\partial\varphi}, \qquad \psi = \frac{\partial W}{\partial J} = \varphi + \varepsilon\frac{\partial W_1}{\partial J}. \tag{3.2.23}$$

- Invert the relations (3.2.23) to obtain I and φ as functions of J and ψ, expressed as series in ε. Express H as a function of the new variables. Choose W_1 in such a way that the coefficient $K_1(J,\psi)$ of order ε is as simple as possible, e.g. depends only on J.
- Repeat this procedure in order to simplify higher order terms.

The aim is to produce, after a certain number of iterations, new variables (J,ψ) in which the Hamiltonian takes a simple form such as

$$K(J,\psi,\varepsilon) = K_0(J) + \varepsilon K_1(J) + \dots + \varepsilon^r K_r(J) + \varepsilon^{r+1} K_{r+1}(J,\psi,\varepsilon), \tag{3.2.24}$$

so that $\dot{J} = \mathcal{O}(\varepsilon^{r+1})$, which means that J is an adiabatic invariant on the time scale $1/\varepsilon^{r+1}$. Ideally, if we were able to eliminate all ψ-dependent terms, we would have produced an integrable system by a change of variables.

Such a transformation does *not* exist in general. But even computing an approximation of the form (3.2.24) to an integrable system by von Zeipel's method, which would be useful for finite but long time scales, turns out to be extremely difficult in practice. The main difficulty is to invert the relations (3.2.23), which requires a lot of computations. Nevertheless, the method was used in the nineteenth century to compute the orbits of the Moon and planets to high order in ε.

3.2.3 Lie–Deprit Series

André Deprit has introduced a different method, allowing to compute simplified Hamiltonians of the form (3.2.24) in a very systematic way, and avoiding the time-consuming inversions. The method has been developed for analytic Hamiltonians of the form

$$H(q,p,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} H_n(q,p). \tag{3.2.25}$$

Instead of simplifying H by successive canonical transformations of the form (3.2.23), which mix old and new variables, only one canonical transformation is carried out, which does not mix the variables. It is defined by a function

$$W(q,p,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} W_n(q,p), \tag{3.2.26}$$

where the W_n have to be determined. The idea is to construct a near-identity transformation by letting ε play the rôle of time: q and p are considered as functions of Q and P via the initial value problem

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\varepsilon} &= \frac{\partial W}{\partial p}(q,p,\varepsilon) && q(0) = Q \\ \frac{\mathrm{d}p}{\mathrm{d}\varepsilon} &= -\frac{\partial W}{\partial q}(q,p,\varepsilon) && p(0) = P. \end{aligned} \tag{3.2.27}$$

For $\varepsilon = 0$, this is simply the identity transformation, while for $0 < \varepsilon \ll 1$, it is a near-identity canonical transformation because it has the structure of a Hamiltonian flow (if $p \cdot \mathrm{d}q - H\,\mathrm{d}t$ and $p \cdot \mathrm{d}q - W\,\mathrm{d}\varepsilon$ are stationary, then $P \cdot \mathrm{d}Q - H\,\mathrm{d}t$ is stationary).

The major advantage is that no functions need to be inverted. The main problem is now, given the W_n, to express $H(q,p,\varepsilon)$ as a function of Q, P and ε. This can be done in a straightforward way, using the notion of a Lie derivative.

Consider first a system with Hamiltonian $H(q,p)$. The *Lie derivative* generated by H is the map

$$L_H : f \mapsto \{f;H\}. \tag{3.2.28}$$

It is convenient to introduce the notations

$$L_H^0(f) = f, \qquad L_H^1(f) = L_H(f), \qquad L_H^k(f) = L_H(L_H^{k-1}(f)), \quad k \geqslant 2. \tag{3.2.29}$$

Then for any analytic function $f(q,p)$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f(q(t),p(t)) &= \{f;H\}(q(t),p(t)) = L_H(f)(q(t),p(t))\\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}f(q(t),p(t)) &= \{\frac{\mathrm{d}}{\mathrm{d}t}f;H\} = L_H(L_H(f)) = L_H^2(f),\end{aligned} \tag{3.2.30}$$

and so on. By Taylor's formula, we thus obtain the relation

$$f(q(t),p(t)) = \sum_{k\geqslant 0}\frac{t^k}{k!}L_H^k(f)(q(0),p(0)), \tag{3.2.31}$$

valid for all t sufficiently small for this series to converge. Symbolically, we can write this relation as

$$f(q(t),p(t)) = \exp\{tL_H\}(f)(q(0),p(0)). \tag{3.2.32}$$

Consider now a case where $f(q,p,t)$ depends explicitly on time. The situation is similar, with the difference that

$$\frac{\mathrm{d}}{\mathrm{d}t}f(q(t),p(t),t) = \{f;H\} + \frac{\partial f}{\partial t}. \tag{3.2.33}$$

Thus is we introduce the notation

$$\Delta_H(f) = \{f;H\} + \frac{\partial f}{\partial t}, \tag{3.2.34}$$

and define Δ_H^k in a similar way as L_H^k, we can write

$$f(q(t),p(t),t) = \sum_{k\geqslant 0}\frac{t^k}{k!}\Delta_H^k(f)(q(0),p(0),0) = \exp\{t\Delta_H\}(f)(q(0),p(0),0). \tag{3.2.35}$$

With these notations, the solution of (3.2.27) can be represented as

$$\begin{aligned}q(Q,P,\varepsilon) &= \exp\{\varepsilon\Delta_W\}(Q)\\ p(Q,P,\varepsilon) &= \exp\{\varepsilon\Delta_W\}(P).\end{aligned} \tag{3.2.36}$$

For a general analytic function $f(q,p,\varepsilon)$, we have

$$F(Q,P,\varepsilon) := f(q(Q,P,\varepsilon),p(Q,P,\varepsilon),\varepsilon) = \exp\{\varepsilon\Delta_W\}(f)(Q,P,0). \tag{3.2.37}$$

The inverse transformation of (3.2.36) is given by

$$\begin{aligned}Q(q,p,\varepsilon) &= \exp\{-\varepsilon\Delta_W\}(q)\\ P(q,p,\varepsilon) &= \exp\{-\varepsilon\Delta_W\}(p).\end{aligned} \tag{3.2.38}$$

We are now prepared to state and prove a general result, allowing to determine the effect of a change of variables defined by (3.2.27) on a function of p and q, such as the Hamiltonian.

Proposition 3.2.5. *Consider an analytic function*

$$f(q,p,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} f_n(q,p). \tag{3.2.39}$$

Let (Q,P) *be new variables defined by* (3.2.27) *with*

$$W(q,p,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} W_n(q,p). \tag{3.2.40}$$

Then the expression of f *in the new variables is given by*

$$F(Q,P,\varepsilon) := f(q(Q,P,\varepsilon), p(Q,P,\varepsilon), \varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} F_n(Q,P), \tag{3.2.41}$$

where the F_n *are determined iteratively as follows. Let* $f_n^0(Q,P) = f_n(Q,P)$*. Define functions* $f_n^k(Q,P)$ *recursively by the relation*

$$f_n^k(Q,P) = f_{n+1}^{k-1}(Q,P) + \sum_{m=0}^{n} \binom{n}{m} \{f_{n-m}^{k-1}; W_m\}(Q,P). \tag{3.2.42}$$

Then $F_n(Q,P) = f_0^n(Q,P)$*.*

Proof: The idea is to compute $\Delta_W^k(f)$ by induction, and then to use (3.2.37). We first observe that

$$\begin{aligned}
\frac{\partial}{\partial\varepsilon} f(Q,P,\varepsilon) &= \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} f_{n+1}(Q,P) \\
L_W f(Q,P,\varepsilon) &= \sum_{n\geqslant 0}\sum_{m\geqslant 0} \frac{\varepsilon^n}{n!}\frac{\varepsilon^m}{m!} \{f_n; W_m\}(Q,P) \\
&= \sum_{k\geqslant 0} \frac{\varepsilon^k}{k!} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} \{f_{n-m}; W_m\}(Q,P).
\end{aligned}$$

This shows that

$$\Delta_W f(Q,P,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} f_n^1(Q,P),$$

where

$$f_n^1(Q,P) = f_{n+1}(Q,P) + \sum_{m=0}^{n} \binom{n}{m} \{f_{n-m}; W_m\}(Q,P).$$

In particular, for $\varepsilon = 0$ we find $F_1(Q,P) = \Delta_W(f)(Q,P,0) = f_0^1(Q,P)$. Now by induction, one obtains in a similar way that

$$\Delta_W^k f(Q,P,\varepsilon) = \sum_{n\geqslant 0} \frac{\varepsilon^n}{n!} f_n^k(Q,P),$$

where

$$f_n^k(Q,P) = f_{n+1}^{k-1}(Q,P) + \sum_{m=0}^{n} \binom{n}{m} \{f_{n-m}^{k-1}; W_m\}(Q,P).$$

Since, for $\varepsilon = 0$, $F_k(Q,P) = \Delta_W^k(f)(Q,P,0) = f_0^k(Q,P)$, the assertion is proved. □

The iterative scheme may by represented by the *Lie triangle*:

$$
\begin{array}{ccccccccccc}
 & & F_0 & & F_1 & & F_2 & & F_3 & & \dots \\
 & & \| & & \| & & \| & & \| & & \\
f_0 & = & f_0^0 & \longrightarrow & f_0^1 & \longrightarrow & f_0^2 & \longrightarrow & f_0^3 & \longrightarrow & \dots \\
 & & & \nearrow & & \nearrow & & \nearrow & & & \\
f_1 & = & f_1^0 & \longrightarrow & f_1^1 & \longrightarrow & f_1^2 & \longrightarrow & \dots & & \\
 & & & \nearrow & & \nearrow & & & & & \\
f_2 & = & f_2^0 & \longrightarrow & f_2^1 & \longrightarrow & \dots & & & & \\
 & & & \nearrow & & & & & & & \\
f_3 & = & f_3^0 & \longrightarrow & \dots & & & & & & \\
 & & & & & & & & & & \\
\dots & & & & & & & & & &
\end{array}
$$

In practice, one proceeds as follows. The Hamiltonian $H(q, p, \varepsilon)$ is expanded in ε to some finite order r, and provides the entries H_n in the left column of the triangle. One can then compute the H_n^k in one diagonal after the other, and the upper row will contain the components K_n of the Hamiltonian in the new variables. Each diagonal depends only on previously computed quantities and one additional term W_n.

The first two relations are

$$
\begin{aligned}
K_0(Q, P) &= H_0^0(Q, P) = H_0(Q, P) \\
K_1(Q, P) &= H_0^1(Q, P) = H_1(Q, P) + \{H_0; W_0\}(Q, P).
\end{aligned} \tag{3.2.43}
$$

The second relation allows us to determine W_0, which we choose in such a way that K_1 is as simple as possible. The second diagonal of the triangle can then be computed by the relations

$$
\begin{aligned}
H_1^1 &= H_2 + \{H_1; W_0\} + \{H_0; W_1\} \\
H_0^2 &= H_1^1 + \{H_0^1; W_0\},
\end{aligned} \tag{3.2.44}
$$

so that we obtain

$$
K_2 = H_2 + 2\{H_1; W_0\} + \{H_0; W_1\} + \{\{H_0; W_0\}; W_0\}. \tag{3.2.45}
$$

Again, this allows us to choose W_1 in such a way as to simplify K_2 as much as possible. This procedure is very systematic, and thus suitable for computer algebra.

Example 3.2.6. Consider again a Hamiltonian as in Example 3.2.4:

$$
H(\varphi_1, \varphi_2, I_1, I_2, \varepsilon) = H_0(I_1) + \sum_{n \geqslant 1} \frac{\varepsilon^n}{n!} H_n(\varphi_1, \varphi_2, I_1, I_2), \tag{3.2.46}
$$

where $\Omega(I_1) = H_0'(I_1) \neq 0$. We would like to determine W in such a way as to eliminate the dependence on φ_1 of the H_n. From (3.2.43) and (3.2.44), we get

$$
\begin{aligned}
K_0 &= H_0 \\
K_1 &= H_1 + \{H_0; W_0\} \\
K_2 &= H_2 + 2\{H_1; W_0\} + \{H_0; W_1\} + \{\{H_0; W_0\}; W_0\}.
\end{aligned} \tag{3.2.47}
$$

Since H_0 depends only on I_1, the second relation can be written as

$$K_1(\psi, J) = H_1(\psi, J) - \Omega(J_1)\frac{\partial W_0}{\partial \psi_1}(\psi, J). \tag{3.2.48}$$

We want to find W_0 periodic in ψ_1 such that K_1 no longer depends on ψ_1. This can be achieved by taking

$$W_0(\psi_1, \psi_2, J_1, J_2) = \frac{1}{\Omega(J_1)} \int_0^{\psi_1} \bigl[H_1(\psi_1', \psi_2, J_1, J_2) - \langle H_1 \rangle(\psi_2, J_1, J_2)\bigr] \, \mathrm{d}\psi_1', \tag{3.2.49}$$

where $\langle H_1 \rangle$ denotes the average of H_1 over ψ_1. As a result, we have

$$K_1(\psi_1, \psi_2, J_1, J_2) = \langle H_1 \rangle(\psi_2, J_1, J_2). \tag{3.2.50}$$

Thus in this particular case, the Lie–Deprit method to first order is identical with the averaging method. Continuing to higher orders, we can produce a new Hamiltonian of the form

$$\begin{aligned} K(\psi_1, \psi_2, J_1, J_2, \varepsilon) = K_0(J_1) &+ \varepsilon K_1(\psi_2, J_1, J_2) + \dots + \varepsilon^r K_r(\psi_2, J_1, J_2) \\ &+ \varepsilon^{r+1} R(\psi_1, \psi_2, J_1, J_2, \varepsilon). \end{aligned} \tag{3.2.51}$$

Writing down the canonical equations, we find in particular that $\dot{J}_1 = \mathcal{O}(\varepsilon^{r+1})$, and thus J_1 is an adiabatic invariant on the time scale $1/\varepsilon^r$.

3.3 Kolmogorov–Arnol'd–Moser Theory

So far, we have discussed a few iterative methods allowing to solve approximately differential equations depending on a small parameter. By r successive changes of variables, the original equation is transformed into a perturbation of order ε^{r+1} of a solvable equation. The solution of the transformed equation remains, on some time scale, ε^{r+1}-close to the solution of its approximation. Going back to original variables, we thus obtain an expansion of the form

$$x(t) = x^0(t) + \varepsilon x^1(t) + \dots + \varepsilon^r x^r(t) + \varepsilon^{r+1} R_{r+1}(t, \varepsilon), \tag{3.3.1}$$

where the functions $x^0, \dots, x^r$ can be computed, and $\|R\|$ can be bounded on some time interval.

For a number of purposes, expansions of the form (3.3.1) are sufficient. If one wants to understand the long-time behaviour of the system, however, they are not of much use. So the question arises whether the limit $r \to \infty$ can be taken, in such a way that the remainder $\varepsilon^{r+1} R_{r+1}$ disappears. In certain cases, it turns out that although the $x^j(t)$ have more or less constant amplitude for a number of j, they ultimately grow very quickly with j, preventing the convergence of the series.

Poincaré already pointed out the difference between a mathematician's definition of convergence of a series, and a more pragmatic one, useful in applications. Consider the two formal series

$$S_1 = \sum_{j \geqslant 0} \frac{(1000)^j}{j!} = 1 + 10^3 + \frac{1}{2} \cdot 10^6 + \frac{1}{6} \cdot 10^9 + \dots = \mathrm{e}^{1000} \tag{3.3.2}$$

$$S_2 = \sum_{j \geqslant 0} \frac{j!}{(1000)^j} = 1 + 10^{-3} + 2 \cdot 10^{-6} + 6 \cdot 10^{-9} + \dots \tag{3.3.3}$$

The first series is convergent, while the second is only formal. However, the fact that S_1 converges is pretty useless in practice since the first thousand terms increase, while the partial sums of the divergent series S_2 hardly change for the first thousand terms.

In a number of cases, it has been shown that the series (3.3.1) diverges for some initial conditions. There are thus two possible approaches to the problem:

1. Admitting that (3.3.1) may diverge for some initial conditions, find the optimal r such that the remainder $\varepsilon^{r+1}R_{r+1}(t,\varepsilon)$ is as small as possible. This approach was first developed by Nekhoroshev and Neishtadt, and we will discuss an example in the last chapter.
2. Find initial conditions for which the series (3.3.1) converges. This was first achieved by Kolmogorov in 1954 for a class of analytic near-integrable Hamiltonians (he did only publish an outline of the proof), and generalized by Arnol'd in 1961. In 1962, Moser obtained a similar result for near-integrable mappings of *finite* differentiability.

Results of the second category are known as Kolmorogov–Arnol'd–Moser (or KAM) theory. They have been considerably developed in the last 40 years, and are still an active area of research.

In the sequel, we shall illustrate the problems and results in a relatively simple case, which can be described by Moser's theorem. After that, we will mention some other results of KAM theory.

3.3.1 Normal Forms and Small Denominators

Let us consider a two-degree-of-freedom Hamiltonian $H(q_1,q_2,p_1,p_2)$, admitting a periodic solution $\gamma(t)$. Our aim is to determine the stability of γ. We will assume for simplicity that q_1 is an angular variable such that

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} > 0 \tag{3.3.4}$$

in a neighbourhood of γ (this can be achieved by a local canonical transformation). Note that Condition (3.3.4) implies that the function H can be inverted with respect to p_1 near γ, i.e., we have

$$p_1 = P(H,q_1,q_2,p_2). \tag{3.3.5}$$

Since H is a constant of the motion, trajectories are contained in a three-dimensional level set of H, which can be parametrized by q_1, q_2 and p_2 (the last coordinate p_1 being determined by (3.3.5)). Condition (3.3.4) also implies that the vector field is transverse to hyperplanes $q_1 =$ *constant*. We can thus take a Poincaré section, say, at $q_1 = 0 \equiv 2\pi$, and consider the Poincaré map

$$\Pi : (q_2,p_2,q_1=0) \mapsto (q_2,p_2,q_1=2\pi), \tag{3.3.6}$$

which is, in effect, two-dimensional, and depends on H as on a parameter. Note, moreover, that

$$\begin{aligned}
\frac{\mathrm{d}q_2}{\mathrm{d}q_1} &= \frac{\partial H/\partial p_2}{\partial H/\partial p_1} = -\frac{\partial P}{\partial p_2} \\
\frac{\mathrm{d}p_2}{\mathrm{d}q_1} &= -\frac{\partial H/\partial q_2}{\partial H/\partial p_1} = \frac{\partial P}{\partial q_2}.
\end{aligned} \tag{3.3.7}$$

P plays the rôle of a (time-dependent) Hamiltonian, which shows in particular that Π is area-preserving.

Let $x = (q_2, p_2)$ and denote by $x^\star$ the coordinates of the intersection of γ with the section $q_1 = 0$. Then $\Pi(x^\star) = x^\star$, and since Π is area-preserving, the Jacobian of Π is equal to 1. There are three possibilities:

1. $\frac{\partial \Pi}{\partial x}(x^\star)$ has real eigenvalues $a_1 \neq \pm 1$ and $a_2 = 1/a_1$. Then $x^\star$ is a hyperbolic saddle point of Π, and thus γ is unstable because nearby orbits are repelled along the unstable manifold.
2. $\frac{\partial \Pi}{\partial x}(x^\star)$ has complex eigenvalues $\mathrm{e}^{\pm 2\pi\,\mathrm{i}\,\theta}$ of unit module, $\theta \notin \mathbb{Z}$, and $x^\star$ is called an *elliptic* equilibrium. This is the case we want to study.
3. $\frac{\partial \Pi}{\partial x}(x^\star)$ has eigenvalues $+1$ or -1, and $x^\star$ is called a *parabolic* equilibrium. This situation arises in bifurcations, we will not discuss it here.

In an appropriate basis, the linearization of Π at $x^\star$ is a rotation of angle $2\pi\theta$. We can introduce a complex variable z for which the Poincaré map becomes

$$z \mapsto \hat{z} = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta} z + g(z, \overline{z}), \tag{3.3.8}$$

where g admits a Taylor expansion of the form

$$g(z, \overline{z}) = \sum_{n+m=2}^{r} g_{nm} z^n \overline{z}^m + \mathcal{O}(|z|^r) \tag{3.3.9}$$

if the original Hamiltonian is of class $\mathcal{C}^r$.

If the term g were absent, the map (3.3.8) would be a rotation, and thus the periodic solution γ would be stable. A natural thing to do is thus try to eliminate g by a change of variables, i.e., compute a *normal form* of (3.3.8). Consider a transformation of the form

$$z = \zeta + c\zeta^j \overline{\zeta}^k, \qquad 2 \leqslant j + k \leqslant r. \tag{3.3.10}$$

Since $g(z, \overline{z}) = g(\zeta, \overline{\zeta}) + \mathcal{O}(|\zeta|^{j+k+1})$, the map (3.3.8) becomes

$$\hat{\zeta} + c\hat{\zeta}^j \overline{\hat{\zeta}}^k = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta} \zeta + c\,\mathrm{e}^{2\pi\,\mathrm{i}\,\theta} \zeta^j \overline{\zeta}^k + g(\zeta, \overline{\zeta}) + \mathcal{O}(|\zeta|^{j+k+1}). \tag{3.3.11}$$

This shows that $\hat{\zeta} = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta} \zeta + \mathcal{O}(|\zeta|^2)$. Substituting this in the term $c\hat{\zeta}^j \overline{\hat{\zeta}}^k$ and grouping like powers of ζ, we get

$$\hat{\zeta} = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta} \zeta + c\bigl(\mathrm{e}^{2\pi\,\mathrm{i}\,\theta} - \mathrm{e}^{2\pi\,\mathrm{i}(j-k)\theta}\bigr)\zeta^j \overline{\zeta}^k + \sum_{n+m=2}^{r} g_{nm} \zeta^n \overline{\zeta}^m + \mathcal{O}(|\zeta|^{j+k+1}). \tag{3.3.12}$$

We see that we can kill the term proportional to $\zeta^j \overline{\zeta}^k$ if we choose

$$c = \frac{g_{jk}}{\mathrm{e}^{2\pi\,\mathrm{i}(j-k)\theta} - \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}} = \frac{g_{jk}\,\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}}{\mathrm{e}^{2\pi\,\mathrm{i}(j-k-1)\theta} - 1}. \tag{3.3.13}$$

The transformation creates new terms, but they are of higher order in $|\zeta|$. We can thus simplify the map (3.3.8) by eliminating first terms of order 2, then terms of order 3, and so on up to order r.

However, the equation (3.3.13) for c can only be solved under the condition

$$\mathrm{e}^{2\pi\,\mathrm{i}(j-k-1)\theta} \neq 1. \tag{3.3.14}$$

Terms $g_{jk} z^j \overline{z}^k$ which violate this condition are called *resonant*. There are different kinds of resonant terms:

- terms for which $k = j-1$ are always resonant, they are of the form $g_{k+1\,k}|z|^{2k}z$;
- if $\theta = p/q$ is rational, terms for which $j-k-1$ is a multiple of q are also resonant, they are of the form $g_{jj+1+nq}|z|^{2j}z^{1+nq}$ (the smaller q, the more such resonant terms exist).

One can show, in fact, that the origin is an unstable fixed point of the map if $\theta = p/q$ for some $q \leqslant 4$. On the other hand, if θ is irrational, the normal form of (3.3.8) is

$$\hat{\zeta} = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\,\zeta + C_1|\zeta|^2\zeta + \dots + C_m|\zeta|^{2m}\zeta + \mathcal{O}(|\zeta|^r), \qquad m = \left[\frac{r}{2} - 1\right], \tag{3.3.15}$$

where $[\cdot]$ denotes the integer part. Such a map is called a *Birkhoff normal form.*

However, even if g is analytic and θ is irrational, it is not clear whether all terms which are not of the form $|\zeta|^{2j}\zeta$ can be eliminated. The reason is that the remainder at some order $j+k+1$ will involve terms depending on c given by (3.3.13), the denominator of which is of the form $\mathrm{e}^{2\pi\,\mathrm{i}\,q\theta} - 1$ for some integer q. Since the numbers $\mathrm{e}^{2\pi\,\mathrm{i}\,q\theta}$ are dense on the unit circle, the quantity $\mathrm{e}^{2\pi\,\mathrm{i}\,q\theta} - 1$ may become arbitrarily small for large q. This is the problem of the *small denominators.*

For now, let us assume that the differentiability r is at least 4 and that

$$\left(\mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\right)^q \neq 1 \qquad\qquad \text{for } q = 1,2,3,4. \tag{3.3.16}$$

Then the Birkhoff normal form is

$$\hat{\zeta} = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}\,\zeta + C_1|\zeta|^2\zeta + \mathcal{O}(|\zeta|^4). \tag{3.3.17}$$

We now introduce polar coordinates $\zeta = I\,\mathrm{e}^{\mathrm{i}\,\varphi}$ (where I will play the rôle of an action variable). In order to find the expression of the map in polar coordinates, we first compute

$$\hat{I}^2 = |\hat{\zeta}|^2 = I^2 + 2\,\mathrm{Re}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)I^4 + \mathcal{O}(I^5) \tag{3.3.18}$$

which determines $\hat{I}$. This allows us to compute

$$\mathrm{e}^{\mathrm{i}\,\hat{\varphi}} = \mathrm{e}^{\mathrm{i}\,\varphi + 2\pi\,\mathrm{i}\,\theta}\big(1 + \mathrm{i}\,\mathrm{Im}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)I^2\big) + \mathcal{O}(I^3). \tag{3.3.19}$$

Taking the logarithm and expanding in I, we find that the map in polar coordinates takes the form

$$\begin{aligned}\hat{\varphi} &= \varphi + 2\pi\theta + \mathrm{Im}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)I^2 + \mathcal{O}(I^3)\\ \hat{I} &= I + \mathrm{Re}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)I^3 + \mathcal{O}(I^4).\end{aligned} \tag{3.3.20}$$

If $\mathrm{Re}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)$ were negative, we could conclude that $\hat{I} < I$ for sufficiently small I, which would imply the asymptotic stability of the orbit. This, however, is incompatible with the fact that the map is area-preserving, since it would imply that the image of a small disc is strictly contained in the disc. For similar reasons, $\mathrm{Re}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1)$ cannot be positive, and we conclude that $\mathrm{Re}(\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1) = 0$. We will write the map (3.3.20) in the form

$$\begin{aligned}\hat{\varphi} &= \varphi + \Omega(I) + f(\varphi, I)\\ \hat{I} &= I + g(\varphi, I),\end{aligned} \tag{3.3.21}$$

where

$$\Omega(I) = 2\pi\theta + \frac{\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1}{\mathrm{i}}I^2, \qquad f = \mathcal{O}(I^3), \qquad g = \mathcal{O}(I^4). \tag{3.3.22}$$

If the original system satisfies higher differentiability and non-resonance conditions, the map in polar coordinates can also be written in the form (3.3.21), with accordingly more terms in Ω and smaller remainders f and g.

The smallness of f and g near $I = 0$ can also be expressed by introducing a small parameter ε and a scaled variable $J = I/\varepsilon$. We will, however, stick to the representation (3.3.21).

3.3.2 Diophantine Numbers

We have seen that if we want to eliminate the remainders from maps of the form (3.3.8), (3.3.17) or (3.3.21), we will produce small denominators of the form $\mathrm{e}^{2\pi\,\mathrm{i}\,q\theta} - 1$ with $q \in \mathbb{Z}_*$. The successive changes of variables can only be expected to converge if these denominators are not too small, which amounts to requiring that θ is badly approximated by rationals. Such numbers are called *Diophantine*.

Definition 3.3.1. *A number $\omega \in \mathbb{R} \setminus \mathbb{Q}$ is called* Diophantine of type (C, ν) *for some real numbers $C > 0$, $\nu \geqslant 1$ if*

$$\Bigl|\omega - \frac{p}{q}\Bigr| \geqslant \frac{C}{|q|^{1+\nu}} \tag{3.3.23}$$

for all relatively prime $(p, q) \in \mathbb{Z} \times \mathbb{Z}_*$. Diophantine numbers of type $(C, 1)$ are said to be of constant type.

Note that Condition (3.3.23) becomes harder to fulfill when C increases or ν decreases. One can, of course, wonder whether Diophantine numbers do exist at all. The answer is yes, for C small enough. Algebraic numbers are examples of Diophantine numbers.

Definition 3.3.2. *An irrational number ω is called* algebraic of order $n \geqslant 2$ *if there exists a polynomial with integer coefficients*

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0, \qquad a_0, \dots, a_n \in \mathbb{Z}\,,\ a_n \neq 0, \tag{3.3.24}$$

such that $P(\omega) = 0$. Algebraic numbers of order 2 are called quadratic.

For instance, $\sqrt{2}$ is a quadratic irrational. The following result, due to Liouville, states that algebraic numbers are indeed Diophantine.

Theorem 3.3.3 (Liouville). *Let ω be an algebraic irrational of order n, and let k, $0 \leqslant k \leqslant n-2$, be such that*

$$P(\omega) = 0, \quad P'(\omega) = 0, \quad \dots \quad P^{(k)}(\omega) = 0, \quad P^{(k+1)}(\omega) \neq 0. \tag{3.3.25}$$

(Note that $P^{(n-1)}(\omega) = n!\omega \neq 0$). Then ω is Diophantine of type $(C, \frac{n}{k+1} - 1)$ for some $C > 0$. In particular, quadratic irrationals are of constant type.

Proof: The zeroes of polynomials being isolated, there exists a $\delta > 0$ such that $P(x) \neq 0$ for $0 < |x - \omega| < \delta$. It is sufficient to check Condition (3.3.23) for $|\omega - p/q| < \delta$, since for other rationals p/q, it is trivially satisfied with $C = \delta$. Since

$$q^n P\Bigl(\frac{p}{q}\Bigr) = \sum_{j=0}^{n} a_j \Bigl(\frac{p}{q}\Bigr)^j q^n \in \mathbb{Z}\,,$$

the fact that $P(p/q) \neq 0$ implies that

$$\Big|q^n P\Big(\frac{p}{q}\Big)\Big| \geqslant 1.$$

On the other hand, Taylor's formula shows that there exists a z between ω and p/q such that

$$P\Big(\frac{p}{q}\Big) = \underbrace{P(\omega) + \sum_{j=1}^{k} \frac{P^{(j)}(\omega)}{j!}\Big(\frac{p}{q} - \omega\Big)^j}_{=0} + \frac{P^{(k+1)}(z)}{(k+1)!}\Big(\frac{p}{q} - \omega\Big)^{k+1}.$$

This shows that there exists a constant M, depending only on P and δ, such that

$$\Big|P\Big(\frac{p}{q}\Big)\Big| \leqslant M\Big|\frac{p}{q} - \omega\Big|^{k+1},$$

and thus

$$\Big|\omega - \frac{p}{q}\Big| \geqslant \Big(\frac{1}{M}\Big|P\Big(\frac{p}{q}\Big)\Big|\Big)^{1/(k+1)} \geqslant \Big(\frac{1}{Mq^n}\Big)^{1/(k+1)}.$$

The result follows with $C = \min\{\delta, M^{-1/(k+1)}\}$. □

Note that $k = 0$ is always possible, so that an algebraic number of order n is always Diophantine of type $(C, n-1)$. This result is certainly not optimal, except for $n = 2$. There are no Diophantine numbers of type (C, ν) for $\nu < 1$, but Roth showed that all algebraic irrationals are Diophantine of type $(C(\varepsilon), 1+\varepsilon)$ for any $\varepsilon > 0$.

There are Diophantine numbers which are not algebraic, in fact Diophantine numbers form a (Cantor) set of positive measure for C small enough. This can be seen by noting that Diophantine numbers in $[0,1]$ are obtained by excluding intervals of length C around 0 and 1, of length $C/2^{1+\nu}$ around $1/2$, of length $C/3^{1+\nu}$ around $1/3$ and $2/3$, and so on. Since there are less than q rationals of the form p/q in $[0,1]$, the total measure of numbers in $[0,1]$ which are *not* Diophantine of type (C,ν) is smaller than

$$\sum_{q\geqslant 1} q\frac{C}{q^{1+\nu}} = \sum_{q\geqslant 1} \frac{C}{q^{\nu}} = C\zeta(\nu), \tag{3.3.26}$$

where the Riemann Zeta Function $\zeta(s) = \sum_{n\geqslant 1} n^{-s}$ is bounded for $s > 1$. Thus Diophantine numbers of type (C,ν) have positive measure for $\nu > 1$ and $C < 1/\zeta(\nu)$. In fact, the upper bound (3.3.26) can be replaced by an exact value, using properties of the number $\phi(q)$ of integers $0 < p < q$ relatively prime with q ($\phi(q)$ is called the *Euler function*).

If ω is a Diophantine irrational of type (C,ν), we can find a lower bound for the small denominator in (3.3.13). Indeed, for all $q \in \mathbb{Z}_*$, we have

$$|\mathrm{e}^{2\pi\,\mathrm{i}\,q\omega} - 1|^2 = 2(1 - \cos(2\pi q\omega)) = 4\sin^2(\pi q\omega). \tag{3.3.27}$$

Let p be the integer closest to $q\omega$. Since $|\sin x| > \frac{2}{\pi}|x|$ for $|x| \leqslant \frac{\pi}{2}$ and $|\sin x|$ is π-periodic in x, we have

$$|\mathrm{e}^{2\pi\,\mathrm{i}\,q\omega} - 1| = 2|\sin(\pi q\omega - \pi p)| \geqslant 4|q\omega - p| \geqslant 4\frac{C}{|q|^{\nu}}. \tag{3.3.28}$$

3.3.3 Moser's Theorem

Let us now return to the map

$$\begin{aligned}\hat{\varphi} &= \varphi + \Omega(I) + f(\varphi, I) \\ \hat{I} &= I + g(\varphi, I).\end{aligned} \tag{3.3.29}$$

If f and g are identically zero, the dynamics is a simple rotation: I is constant while φ increases at each iteration by an amount depending only on I. The curves $I =$ *constant* are invariant curves of the map (which are closed since φ is periodic). If this map were structurally stable, we should be able to find a change of variables $(\varphi, I) \mapsto (\psi, J)$ such that the dynamics of the perturbed system (3.3.29) reduces to a rotation for (ψ, J). This is not possible in general, but Moser's theorem asserts that it is possible for some initial conditions, living on a Cantor set.

Theorem 3.3.4 (Moser). *Assume that the area-preserving map* (3.3.29) *is* $\mathcal{C}^r$, $r \geqslant 4$, *in a strip* $a \leqslant I \leqslant b$. *Assume that* $\Omega(I)$ *satisfies the* twist condition

$$\frac{\mathrm{d}\Omega}{\mathrm{d}I} \geqslant W > 0 \qquad \text{for } a \leqslant I \leqslant b. \tag{3.3.30}$$

We introduce the $\mathcal{C}^r$*-norm*

$$\|f\|_{\mathcal{C}^r} = \sup_{\varphi, a \leqslant I \leqslant b} \max_{i+j \leqslant r} \left| \frac{\partial^{i+j} f}{\partial \varphi^i \partial I^j} \right|. \tag{3.3.31}$$

Then for every $\delta > 0$, *there exists a number* $\varepsilon > 0$, *depending on* δ *and* r, *with the following property. If* $\omega \in [\Omega(a) + C, \Omega(b) - C]$ *is Diophantine of type* (C, ν) *for some* $\nu \in (1, \frac{r-1}{2})$, $C > 0$, *and* $\|f\|_{\mathcal{C}^r} + \|g\|_{\mathcal{C}^r} < \varepsilon W C^2$, *then the map* (3.3.29) *admits an invariant curve of the form*

$$\begin{aligned}\varphi &= \psi + u(\psi, \omega) \\ I &= \Omega^{-1}(2\pi\omega) + v(\psi, \omega)\end{aligned} \qquad 0 \leqslant \psi \leqslant 2\pi, \tag{3.3.32}$$

where u *and* v *are* 2π*-periodic in* ψ *and differentiable, with*

$$\|u\|_{\mathcal{C}^1} + \|v\|_{\mathcal{C}^1} < \delta. \tag{3.3.33}$$

The dynamics on this invariant curve is given by the circle map

$$\psi \mapsto \hat{\psi} = \psi + 2\pi\omega. \tag{3.3.34}$$

(ω is called the rotation number *of the invariant curve).*

In fact, it is not necessary to require that the map be area-preserving. It suffices to satisfy the weaker *intersection property*: every curve sufficiently close to $I =$ *constant* should intersect its image. The initial result published by Moser required a differentiability $r \geqslant 333$, and this condition was subsequently weakened by Rüssman, Douady and Herman. In fact, the condition $r > 3$ is sufficient, meaning that the map should be $\mathcal{C}^3$ and satisfy a Hölder condition.

Note that larger values of r allow for larger values of ν, and thus there will be more invariant curves for a given perturbation size ε. Also, C can be taken larger, so that invariant curves subsist for larger perturbations.

We will now give an outline of the proof, and postpone a more detailed discussion of the consequences of Moser's theorem to the next subsection. In this outline, we will not try to establish the optimal relations between ε, δ, r, C and τ, but only give an idea how the functions u and v can be constructed under suitable hypotheses.

Some ideas of the proof of Moser's theorem.

1. **Preliminary transformation:** First note that the average $\langle f\rangle(I)$ can be assumed to vanish, since otherwise we can include this average into Ω. Area preservation implies that if f and g are of order ε, then the average of g is of order ε^2 (to see this, compute the Jacobian of the map (3.3.29) and average over φ). For simplicity, we will assume that g also has average zero, although this assumption is not necessary.
   Since $\Omega'(I) \neq 0$, we can simplify the map by using the variable $\Omega(I)$ instead of I. Then g will be replaced by $\Omega(I+g)-\Omega(I)$. To keep notations simple, we do not introduce new variables, and write the system in the form
$$\begin{aligned}\hat\varphi &= \varphi + I + f(\varphi, I)\\ \hat I &= I + g(\varphi, I),\end{aligned}$$
   where f and g satisfy the former bounds with $W=1$.
2. **A difference equation:** We will need to solve difference equations of the form
$$(\mathcal{L}_\omega u)(\psi) := u(\psi+2\pi\omega) - u(\psi) = h(\psi),$$
   where h is a given 2π-periodic function with zero average. We will denote the solution u of this equation with zero average by $u = \mathcal{L}_\omega^{-1}h$.
   Consider the Fourier series of h,
$$h(\psi) = \sum_{k\in\mathbb{Z}_*} h_k\,\mathrm{e}^{\mathrm{i}\,k\psi}, \qquad\qquad h_k = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\,k\psi}\,h(\psi)\,\mathrm{d}\psi.$$
   Then the solution of the difference equation is given by
$$u(\psi) = (\mathcal{L}_\omega^{-1}h)(\psi) = \sum_{k\in\mathbb{Z}_*}\frac{h_k}{\mathrm{e}^{2\pi\,\mathrm{i}\,k\omega}-1}\,\mathrm{e}^{\mathrm{i}\,k\psi}$$
   At this point, the small denominators have shown up once again. We will need to bound the norm of u, as a function of the norm of h. Note that if $h\in\mathcal{C}^r$, then integration by parts yields
$$h_k = \frac{1}{2\pi\,\mathrm{i}\,k}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\,k\psi}\,h'(\psi)\,\mathrm{d}\psi = \dots = \frac{1}{2\pi(\mathrm{i}\,k)^r}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\,k\psi}\,h^{(r)}(\psi)\,\mathrm{d}\psi,$$
   and thus $|h_k| \leqslant |k|^{-r}\|h\|_{\mathcal{C}^r}$. Using the property (3.3.28) of Diophantine numbers, we obtain
$$\|\mathcal{L}_\omega^{-1}h\|_{\mathcal{C}^0} = \sup_\psi|u(\psi)| \leqslant \sum_{k\in\mathbb{Z}_*}\frac{\|h\|_{\mathcal{C}^r}}{|k|^r}\frac{|k|^\nu}{4C} \leqslant \frac{\zeta(r-\nu)}{2C}\|h\|_{\mathcal{C}^r},$$
   provided $r>\nu$. Note that a larger differentiability r allows for larger values of ν, and thus a larger set of ω's. The effect of the small denominators is now merely that the factor $\zeta(r-\nu)/2C$ may be large, but it is a constant in the problem. Note that under stronger assumptions on $r-\nu$, we can bound higher derivatives of u.
3. **Change of variables:** We would like to find a change of variables
$$\begin{aligned}\varphi &= \psi + u(\psi,\omega)\\ I &= 2\pi\omega + v(\psi,\omega)\end{aligned}$$

transforming the map into

$$\begin{aligned}\hat{\psi} &= \psi + 2\pi\omega \\ \hat{\omega} &= \omega.\end{aligned}$$

Inserting this into our map $(\varphi, I) \mapsto (\hat{\varphi}, \hat{I})$, we find

$$\hat{I} = 2\pi\hat{\omega} + v(\hat{\psi}, \hat{\omega}) = 2\pi\omega + v(\psi, \omega) + g(\psi + u(\psi, \omega), \omega + v(\psi, \omega)).$$

We now obtain a condition for v by requiring that $\hat{\psi} = \psi + 2\pi\omega$ and $\hat{\omega} = \omega$. Proceeding in a similar way with the equation for φ, we end up with the system

$$\begin{aligned}v(\psi + 2\pi\omega, \omega) - v(\psi, \omega) &= g(\psi + u(\psi, \omega), \omega + v(\psi, \omega)) \\ u(\psi + 2\pi\omega, \omega) - u(\psi, \omega) &= v(\psi, \omega) + f(\psi + u(\psi, \omega), \omega + v(\psi, \omega)).\end{aligned}$$

These functional equations are very hard to solve, but we may try to find a solution by iterations. Recall that u and v are expected to be small. As a first approximation, we may neglect the terms u and v appearing in the argument of f and g, so that we are reduced to the difference equation discussed above. Thus we use as a first approximation

$$\begin{aligned}v_0(\psi, \omega) &= \mathcal{L}_\omega^{-1} g(\psi, \omega) \\ u_0(\psi, \omega) &= \mathcal{L}_\omega^{-1} \big[f(\psi, \omega) + v_0(\psi, \omega) \big].\end{aligned}$$

Note that since $\|g\|_{\mathcal{C}^r}$ and $\|f\|_{\mathcal{C}^r}$ are of order εC^2, our bound on $\mathcal{L}_\omega^{-1}$ implies that $\|v_0\|_{\mathcal{C}^0}$ is of order εC and $\|u_0\|_{\mathcal{C}^0}$ is of order ε.
Our aim will now be to construct, by an iterative method, a solution of the functional equations which is close to (u_0, v_0).

4. **Newton's method:** Assume for the moment that we are given a function $F : \mathbb{R} \to \mathbb{R}$ and that we want to solve the equation $F(x) = 0$. Assume further that we know an approximation x_0 of the solution. By Taylor's formula,

$$F(x_0 + \Delta) = F(x_0) + F'(x_0)\Delta + \mathcal{O}(\Delta^2),$$

which vanishes for $\Delta \cong -F'(x_0)^{-1}F(x_0)$. We thus expect that

$$x_1 = T(x_0) \coloneqq x_0 - F'(x_0)^{-1}F(x_0)$$

is a better approximation of the solution. Under appropriate conditions, the sequence of $x_n = T^n(x_0)$ should converge to this solution: This is simply Newton's method. Applying Taylor's formula to second order shows that

$$\begin{aligned}F(x_1) &= F(x_0) + F'(x_0)(x_1 - x_0) + \frac{1}{2}F''(z)(x_1 - x_0)^2 \\ &= \frac{1}{2}F''(z)\big(-F'(x_0)^{-1}F(x_0)\big)^2,\end{aligned}$$

where z lies between x_0 and x_1. Thus if $|F'| \geqslant C$ and $|F''| < K$, we get

$$|F(x_1)| \leqslant \frac{K}{2C^2}|F(x_0)|^2.$$

This behaviour is called *quadratic convergence*. Indeed, if M_0 is a small enough positive number, then the iterates of the map

$$M_{n+1} = \frac{K}{2C^2} M_n^2$$

converge to zero very fast. In fact, we can write $M_n = (2C^2/K)\lambda_n$, where

$$\lambda_{n+1} = \lambda_n^2 \qquad \Rightarrow \qquad \lambda_n = \lambda_0^{2^n},$$

which converges to zero faster than exponentially, provided $\lambda_0 < 1$ (that is, provided $|F(x_0)| < 2C^2/K$). Moreover,

$$|x_n| \leqslant |x_0| + \sum_{j=0}^{n-1} \frac{M_j}{C} = |x_0| + \frac{2C}{K} \sum_{j=0}^{n-1} \lambda_0^{2^n},$$

which converges faster than a geometric series. This fast convergence can overcome the problem of small divisors.

5. **Solving the functional equation:** Let us return to the functional equation. From now on, we will proceed very heuristically, so that what follows should not be taken too literally. We will consider a simplified equation

$$(\mathcal{F}_\omega v)(\psi,\omega) := (\mathcal{L}_\omega v)(\psi,\omega) - g(\psi,\omega + v(\psi,\omega)) = 0.$$

As a first approximation, we use $v_0(\psi,\omega) = \mathcal{L}_\omega^{-1} g(\psi,\omega)$. Then $\|v_0\|_{\mathcal{C}^0}$ is bounded by $(M/C)\|g\|_{\mathcal{C}^r}$, where $M = \zeta(r-\nu)/2$ and C is the constant of the Diophantine condition. This shows that

$$(\mathcal{F}_\omega v_0)(\psi,\omega) = g(\psi,\omega) - g(\psi,\omega + v_0(\psi,\omega))$$

is of order $\|g\|_{\mathcal{C}^1}\|v_0\|_{\mathcal{C}^0} \leqslant (M/C)\|g\|_{\mathcal{C}^r}^2$. Since $\mathcal{F}_\omega$ is a small perturbation of $\mathcal{L}_\omega$, $(\mathcal{F}_\omega')^{-1}$ should also be of the order M/C (here $\mathcal{F}_\omega'$ denotes the Fréchet derivative). If K denotes a bound for $\mathcal{F}_\omega''$, the condition for the convergence of Newton's method thus becomes $\|g\|_{\mathcal{C}^r}^2 < (2/K)(C/M)^3$.

We now construct a sequence of functions v_n by iterations of Newton's method. Our estimate on $|x_n|$ indicates that $\|v_n - v_0\|_{\mathcal{C}^0}$ can be bounded, independently of n, by a constant times $(M/C)^2\|g\|_{\mathcal{C}^r}^2$, which can be made as small as we like by choosing $\|g\|_{\mathcal{C}^r}$ sufficiently small. Moreover, by an argument of uniform convergence, the v_n should have a continuous limit. So it seems plausible that for any prescribed δ, we can find a continuous solution v of the functional equation such that $\|v\|_{\mathcal{C}^0} \leqslant \delta$, provided $\|g\|_{\mathcal{C}^r}$ is smaller than some function of δ and C.

There are, however, several difficulties that we brushed under the carpet. A first problem is to invert $\mathcal{F}_\omega'$ in a sufficiently large domain. Since this is rather difficult, one replaces this inverse by an approximation of the inverse. More serious is the problem of loss of derivatives. Indeed, we only estimated the $\mathcal{C}^0$-norm of $\mathcal{L}_\omega^{-1}(h)$ as a function of the $\mathcal{C}^r$-norm of h, so that we are in trouble when iterating Newton's method. To counteract this problem, one does not apply $\mathcal{L}_\omega^{-1}$ directly to a $\mathcal{C}^r$ function, but to an approximation of this function which has more continuous derivatives. Thus the iterative procedure will not converge as quickly as Newton's method, but it is still possible to make it converge fast enough to counteract the effect of small denominators. Finally, one has to check that the approximations (u_n, v_n) *and* their derivatives converge uniformly. □

3.3.4 Invariant Curves and Periodic Orbits

Let us now examine some consequences of Moser's theorem. A first important consequence it that we have obtained sufficient conditions for the stability of elliptic fixed points.

Corollary 3.3.5. *Let $x^\star$ be an elliptic fixed point of a $\mathcal{C}^4$ area-preserving map Π of the plane. Assume that the Jacobian matrix $\frac{\partial \Pi}{\partial x}(x^\star)$ has eigenvalues $a = \mathrm{e}^{2\pi\,\mathrm{i}\,\theta}$ and $\overline{a}$ such that*

$$|a| = 1 \qquad \text{and} \qquad a^q \neq 1 \qquad \text{for } q = 1,2,3,4. \tag{3.3.35}$$

Assume further that the coefficient C_1 in the Birkhoff normal form $\hat{\zeta} = a\zeta + C_1|\zeta|^2\zeta + \mathcal{O}(|\zeta|^4)$ (c.f. (3.3.17)) is different from zero. Then $x^\star$ is stable, in fact there exists a neighbourhood of $x^\star$ which is invariant under Π.

PROOF: We already showed that the dynamics near $x^\star$ can be described by a map of the form

$$\begin{aligned}\hat{\varphi} &= \varphi + \Omega(I) + f(\varphi, I)\\ \hat{I} &= I + g(\varphi, I),\end{aligned}$$

with

$$\Omega(I) = 2\pi\theta + \frac{\mathrm{e}^{-2\pi\,\mathrm{i}\,\theta}\,C_1}{\mathrm{i}} I^2, \qquad f = \mathcal{O}(I^3), \qquad g = \mathcal{O}(I^4).$$

We should first note that due to the singularity of polar coordinates at the origin, f may only be $\mathcal{C}^3$ at $I = 0$. However, it is $\mathcal{C}^4$ everywhere else in a neighbourhood of $I = 0$ because the map $\zeta \mapsto \hat{\zeta}$ is $\mathcal{C}^4$ and the transformation to polar coordinates is smooth except at the origin. For instance, we might have $f(I,\varphi) = I^{7/2}\sin\varphi$.

We consider now the map in a small strip of the form $\varepsilon \leqslant I \leqslant 2\varepsilon$ (which is a small annulus in original variables). Since we exclude the point $I = 0$, f and g are $\mathcal{C}^4$ in this domain. Replacing I by εJ, we obtain a new map (in the domain $1 \leqslant J \leqslant 2$)

$$\begin{aligned}\hat{\varphi} &= \varphi + \Omega(\varepsilon J) + \tilde{f}(\varphi, J) & \qquad \tilde{f}(\varphi, J) &= f(\varphi, \varepsilon J)\\ \hat{J} &= J + \tilde{g}(\varphi, J) & \tilde{g}(\varphi, J) &= \frac{1}{\varepsilon} g(\varphi, \varepsilon J).\end{aligned}$$

It is easy to check that $\|\tilde{f}\|_{\mathcal{C}^4}$ and $\|\tilde{g}\|_{\mathcal{C}^4}$ are $\mathcal{O}(\varepsilon^3)$. Moreover,

$$\left|\frac{\mathrm{d}}{\mathrm{d}J}\Omega(\varepsilon J)\right| = 2\varepsilon^2|C_1|J \geqslant 2\varepsilon^3|C_1|.$$

Although we did not discuss that point in detail, Moser's theorem remains true if W is a small parameter. Thus the theorem yields the existence of invariant curves in the strip, provided we take ε small enough (i.e., we magnify a sufficiently small neighbourhood of the elliptic point). Since the map leaves invariant $x^\star$ and a curve surrounding $x^\star$, continuity implies that the interior of that curve is invariant. □

Note that Condition (3.3.35) only excludes the strongest resonances $a = \pm 1$, $a = \pm\,\mathrm{i}$ and $a = \mathrm{e}^{\pm 2\pi\,\mathrm{i}/3}$. The result remains true if the argument of a is another rational multiple of 2π because the condition on C_1 implies the twist condition on Ω. The twist condition in turn ensures that even if the angle of rotation is rational at $x^\star$, is will change, and thus go through Diophantine values, as one moves away from $x^\star$.

Let us now illustrate some further properties of maps of the form (3.3.29), by considering the well-studied *standard map*

$$\begin{aligned}\hat{\varphi} &= \varphi + I + \varepsilon \sin\varphi \\ \hat{I} &= I + \varepsilon \sin\varphi.\end{aligned} \tag{3.3.36}$$

Here φ is again an angle (modulo 2π), so that the phase space has the topology of a cylinder. However, for some purposes it is useful to consider φ as real. Then one would define, for instance, an orbit of period q as an orbit such that the point (φ, I) is mapped to $(\varphi + 2\pi p, I)$, $p \in \mathbb{Z}$, after q iterations. Note that the standard map (3.3.36) is area-preserving and satisfies the *twist condition*

$$\frac{\partial\hat{\varphi}}{\partial I} > 0. \tag{3.3.37}$$

General maps satisfying this condition are called *twist maps*.

If $\varepsilon = 0$, I is constant and φ increases at each iteration by I. Since φ is an angle, we can distinguish between two cases:

- if $I/2\pi = p/q$ is rational, then any orbit starting on a point of the form (φ, I) is periodic of period q, and makes p complete rotations after these q iterations;
- if $I/2\pi$ is irrational, then the orbit of (φ, I) never returns to the same point, but fills the curve of constant I in a dense way; such orbits are called *quasiperiodic*.

If $\varepsilon > 0$, Moser's theorem gives us sufficient conditions for the existence of an invariant curve. Since, in the case of the standard map, $\|f\|_{\mathcal{C}^r} = \|g\|_{\mathcal{C}^r} = 1$ for all r, we obtain the existence of an invariant curve with parametric equation

$$\begin{aligned}\varphi &= \psi + u(\psi, \omega) \\ I &= 2\pi\omega + v(\psi, \omega)\end{aligned} \qquad 0 \leqslant \psi \leqslant 2\pi \tag{3.3.38}$$

for every Diophantine ω of type (C, ν), provided $\varepsilon \leqslant \varepsilon_0 C^2$ (where ε_0 is an absolute constant that we called ε in the theorem). The $\mathcal{C}_1$-norms of u and v are bounded by some function $\delta(\varepsilon)$, in fact they are of order ε. Invariant curves of the form (3.3.38), which encircle the cylinder, are called *rotational invariant curves*.

Fig. 3.1 shows that there are indeed many invariant curves for small ε. As ε grows, their number decreases, and the last one is found to disappear for $\varepsilon = 0.97\dots$. This curve is given by (3.3.38) with $\omega = (\sqrt{5} - 1)/2$ equal to the *golden mean*, a quadratic irrational which is particularly hard to approximate by rationals – there is a deep connection between this fact and the properties of continued fractions. For ε increasing beyond 1, the orbits become more and more disordered, and although elliptic orbits still exist, their domains of stability tend to shrink.

We know by Moser's theorem that the dynamics on the invariant curve (3.3.38) is given by $\psi \mapsto \psi + 2\pi\omega$. Thus the j^{th} iterate of (φ_0, I_0) has coordinates

$$\begin{aligned}\varphi_j &= \psi_0 + 2\pi j\omega + u(\psi_0 + 2\pi j\omega, \omega) \\ I_j &= 2\pi\omega + v(\psi_0 + 2\pi j\omega, \omega).\end{aligned} \tag{3.3.39}$$

One can associate with any orbit $\{(\varphi_n, I_n) = \Pi^n(\varphi_0, I_0)\}_{n\in\mathbb{Z}}$ a *rotation number*

$$\Omega = \frac{1}{2\pi} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (\varphi_{j+1} - \varphi_j). \tag{3.3.40}$$

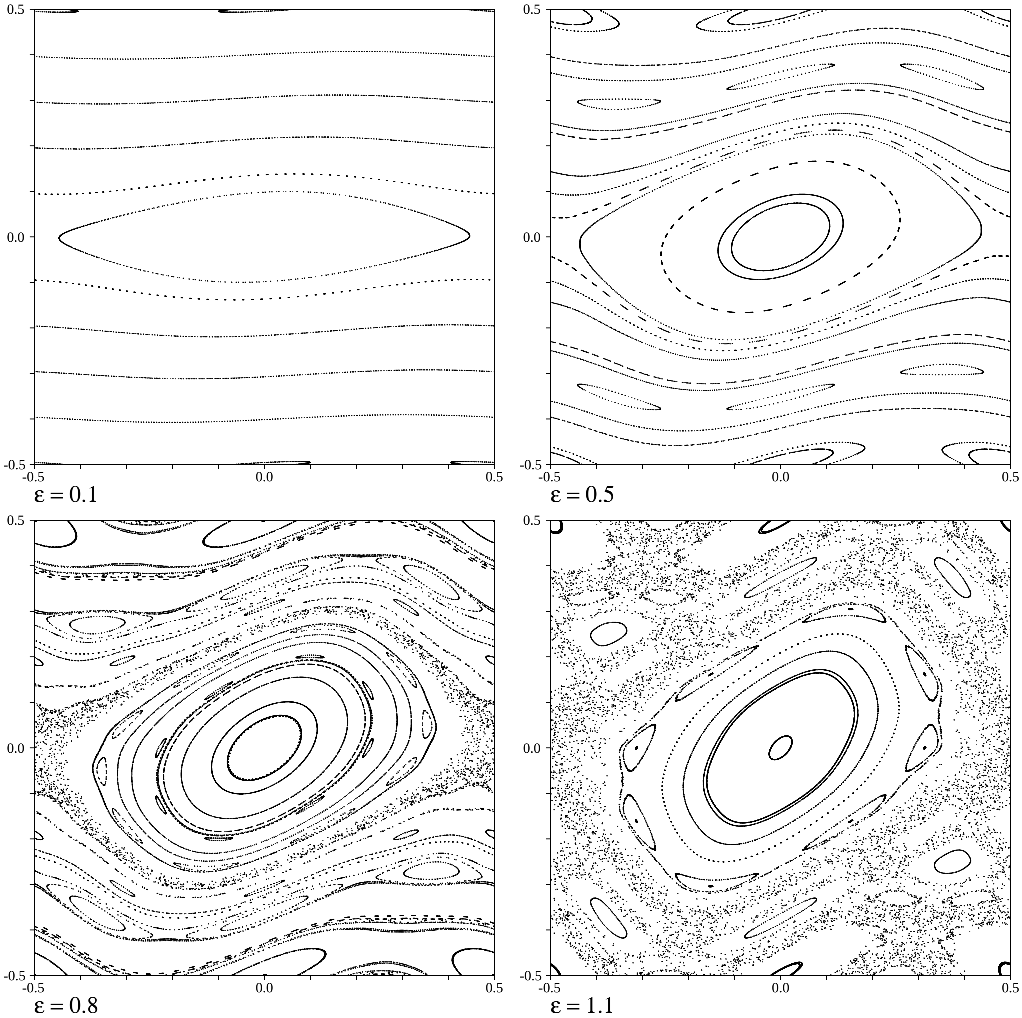

FIGURE 3.1. Phase portraits of the standard map, obtained by representing several orbits with different initial conditions, for increasing values of the perturbation ε.

There is a slight ambiguity because of the angular nature of φ, which implies that Ω is defined up to addition of an integer. For the standard map, we can also define

$$\Omega = \frac{1}{2\pi} \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} (I_j + \varepsilon \sin\varphi_j). \tag{3.3.41}$$

For $\varepsilon = 0$, we have $\Omega = I/2\pi$. To compute the rotation number of the invariant curve (3.3.38), we thus have to average $v(\psi_0 + 2\pi j\omega, \omega)$ over j. Since v is 2π-periodic in its first argument, we use the fact that for each Fourier component $v_k(\omega)\,\mathrm{e}^{\mathrm{i}\,k\psi}$ of v,

$$\sum_{j=1}^{n} v_k(\omega)\,\mathrm{e}^{\mathrm{i}\,k(\psi_0+2\pi j\omega)} = v_k(\omega)\,\mathrm{e}^{\mathrm{i}\,k\psi_0}\,\frac{1-\mathrm{e}^{2\pi\,\mathrm{i}(n-1)k\omega}}{1-\mathrm{e}^{2\pi\,\mathrm{i}\,k\omega}}, \tag{3.3.42}$$

which is bounded uniformly in n. We conclude that the average of $v(\psi_0+2\pi j\omega, \omega)$ vanishes in the limit $n \to \infty$. Hence the rotation number Ω of any orbit on the invariant curve (3.3.38) is ω. The orbits of a twist map can be labelled by their rotation number, and

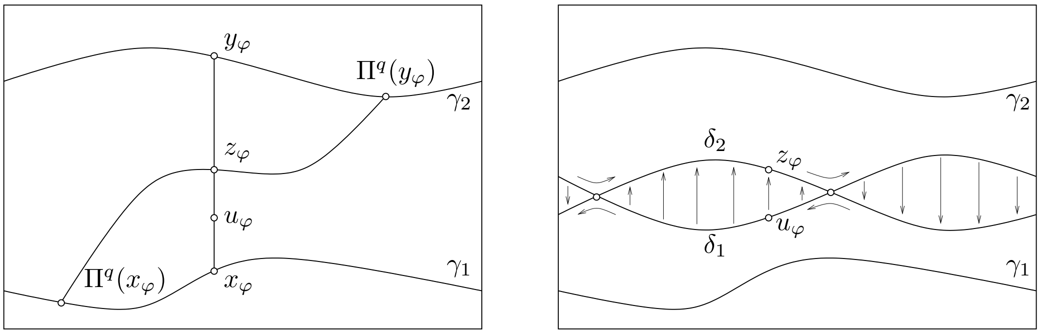

FIGURE 3.2. Construction of periodic orbits with rational rotation number p/q between two invariant curves γ_1 and γ_2. Π^q maps points u_φ on δ_1 vertically to points z_φ on δ_2. Their intersections are thus fixed points of Π^q. The arrows give a hint that one point is elliptic and the other one hyperbolic.

Moser's theorem asserts that orbits with Diophantine rotation number belong to invariant curves for sufficiently small perturbation size ε.

Let us briefly describe other kinds of orbits appearing on Fig. 3.1, some of which are well understood. In particular, we have the following result on existence of periodic orbits.

Theorem 3.3.6 (Poincaré–Birkhoff). *Let* Π *be an area-preserving twist map admitting two rotational invariant curves* γ_1 *and* γ_2*, with respective rotation numbers* $\omega_1 < \omega_2$*. For every rational number* $p/q \in (\omega_1, \omega_2)$*, there exist at least two periodic orbits of period* q*, with rotation number* p/q*, and contained in the domain between* γ_1 *and* γ_2*.*

PROOF: Consider two points $x_\varphi = (\varphi, I_1) \in \gamma_1$ and $y_\varphi = (\varphi, I_2) \in \gamma_2$ (Fig. 3.2). Since $q\omega_1 < p < q\omega_2$, the iterate $\Pi^q(x_\varphi)$ has turned by an angle smaller than $2\pi p$, while $\Pi^q(y_\varphi)$ has turned by an angle larger than $2\pi p$. Thus the vertical line through x_φ and y_φ must cross its image under Π^q at least once, say in z_φ. The point z_φ is not necessarily a fixed point of Π^q, but by construction, it is the image of a point u_φ with the same φ-coordinate.

Consider now the curves $\delta_1 = \{u_\varphi \colon 0 \leqslant \varphi \leqslant 2\pi\}$ and $\delta_2 = \{z_\varphi \colon 0 \leqslant \varphi \leqslant 2\pi\}$. Then $\Pi^q(\delta_1) = \delta_2$, and the φ-coordinates of all points of δ_1 are rotated by an angle $2\pi p$ (i.e., they move vertically). Since the map is area-preserving, the area between γ_1 and δ_1 must be the same as the area between γ_1 and δ_2. Thus δ_1 and δ_2 must intersect at least twice. These intersections are fixed points of Π^q, and thus belong to two orbits of period q of Π. Since each point makes p rotations during q iterations, the rotation number is p/q. □

The assumptions of the theorem can be weakened. There exist more powerful techniques to find periodic orbits, for instance variational ones, where one looks for stationary points of an action functional. It can be shown that if the periodic orbits described in the theorem are non-degenerate, then one of them is elliptic and another one hyperbolic. Chains of elliptic and hyperbolic points with rotation numbers 0, 1/2, 1/3 and 1/4 are easily located on Fig. 3.1 for $\varepsilon = 0.8$.

As we have shown, elliptic orbits are generically stable. The dynamics in their neighbourhood can again be described, in polar coordinates, by a twist map, so that the same layered structure of invariant curves and elliptic and hyperbolic orbits exists around each elliptic orbit. This self-similar structure is most apparent for the standard map when ε is close to 1.

Hyperbolic orbits, on the other hand, are often surrounded by chaotic orbits. This is related to the fact that the stable and unstable manifolds of hyperbolic orbits of given rotation number tend to intersect, producing a so-called Smale horseshoe with complicated dynamics.

The phase portraits on Fig. 3.1 are quite typical for Poincaré sections of two-degrees-of-freedom Hamiltonians. They also illustrate the limits of the method of averaging. Assuming that such a phase portrait represents the Poincaré map of a Hamiltonian system with one fast variable, the averaged system would be integrable, and thus its Poincaré section would consist only of invariant curves and periodic orbits. The chaotic orbits of the original system are smeared out by the averaging transformation.

Many extensions of Moser's theorem exist, for instance to multidimensional symplectic maps. There has also been major progress in "converse KAM" theory, which describes when and how invariant curves or tori break up.

3.3.5 Perturbed Integrable Hamiltonian Systems

The other class of KAM theorems applies to Hamiltonian systems of the form

$$H(I,\varphi) = H_0(I) + \varepsilon H_1(I,\varphi,\varepsilon) \tag{3.3.43}$$

with n degrees of freedom. Let us first explain where the small denominators come from. Assume that we want to eliminate the φ-dependence of the first-order term $H_1(I,\varphi,0)$ by the Lie–Deprit method. As we have seen, the new first order term after a canonical transformation with generator W_0 will be

$$\begin{aligned} K_1(J,\psi) &= H_1(J,\psi) + \{H_0;W_0\}(J,\psi) \\ &= H_1(J,\psi) - \sum_{j=1}^{n} \Omega_j(J)\frac{\partial W_0}{\partial \psi_j}(J,\psi), \qquad\qquad \Omega_j(J) := \frac{\partial H_0}{\partial I_j}(J). \end{aligned} \tag{3.3.44}$$

Let us write H_1 and W_0 in Fourier series

$$H_1(J,\psi) = \sum_{k\in\mathbb{Z}^n} H_{1k}(J)\,\mathrm{e}^{\mathrm{i}\,k\cdot\psi}, \qquad\qquad W_0(J,\psi) = \sum_{k\in\mathbb{Z}^n} W_{0k}(J)\,\mathrm{e}^{\mathrm{i}\,k\cdot\psi}. \tag{3.3.45}$$

Then we have to solve, for each $k \in \mathbb{Z}^n \setminus \{0,\dots,0\}$, an equation of the form

$$\sum_{j=1}^{n} \Omega_j(J) W_{0k}(J)\,\mathrm{i}\,k_j\,\mathrm{e}^{\mathrm{i}\,k\cdot\psi} = H_{1k}(J)\,\mathrm{e}^{\mathrm{i}\,k\cdot\psi}. \tag{3.3.46}$$

We thus take

$$W_{0k}(J) = \frac{H_{1k}(J)}{\mathrm{i}\,k\cdot\Omega(J)} \qquad\qquad \forall k \in \mathbb{Z}^n \setminus \{0,\dots,0\}. \tag{3.3.47}$$

If $k\cdot\Omega(J) = \sum_{j=1}^{n} k_j\Omega_j(J) = 0$ for some k, then the corresponding term of the Fourier series is *resonant* and cannot be eliminated. But even if the $\Omega_j(J)$, which are the frequencies of the unperturbed system, are incommensurate, the term $k\cdot\Omega(J)$ may become arbitrarily small, unless we impose a Diophantine condition on the $\Omega_j(J)$. The following theorem by Arnol'd gives sufficient conditions for the existence of a change of variables, defined on some Cantor set with Diophantine frequencies, which eliminates the dependence of H on the angles. It thus implies the existence of invariant tori on which the motion is quasiperiodic with Diophantine frequencies.

Theorem 3.3.7 (Arnol'd). *Assume the Hamiltonian* (3.3.43) *is analytic, and satisfies the non-degeneracy condition*

$$\left|\det \frac{\partial^2 H_0}{\partial I^2}(I)\right| \geqslant W > 0 \tag{3.3.48}$$

in a neighbourhood of the torus $I = I_0$. Let $\omega = \Omega(I_0) \in \mathbb{R}^n$ satisfy the Diophantine condition

$$\left|\omega \cdot k\right| \geqslant \frac{C}{|k|^\nu} \qquad \forall k \in \mathbb{Z}^n \setminus \{0, \dots, 0\}. \tag{3.3.49}$$

where $|k| = |k_1| + \dots + |k_n|$. Then, if ε is sufficiently small, the Hamiltonian system (3.3.43) *admits a quasiperiodic solution with frequency ω, i.e., this solution can be written in the form*

$$(q(t), p(t)) = F(\omega_1 t, \dots, \omega_n t), \tag{3.3.50}$$

where F is 2π-periodic in all its arguments. This solution lies on an analytic torus, which it fills densely. The distance between this torus and the unperturbed torus $I = I_0, \varphi \in \mathbb{T}^n$ goes to zero as $\varepsilon \to 0$.

In this case, the invariant tori of the perturbed system can be labelled by the frequency vector ω, which plays an analogous rôle as the rotation number for twist maps (since the Diophantine condition only implies ratios of components of ω, it does not matter whether we divide ω by 2π or not).

Let us now examine the consequences of this result on the stability of perturbed integrable systems. If the number of degrees of freedom is $n = 2$, the manifold of constant H is three-dimensional, and contains two-dimensional tori for ε small enough (the intersections of these tori with a two-dimensional Poincaré section are invariant circles). Since these tori act as barriers, and there are many of them, all orbits, even those which do not start on an invariant torus, will be confined to a relatively small portion of phase space, close to an invariant torus of the unperturbed system. Thus the system is stable with respect to small perturbations.

The situation is different for $n > 2$. For instance, if $n = 3$, the level sets of H are five-dimensional and the invariant tori are three-dimensional, hence they have codimension 2. This means that tori no longer act as barriers, and trajectories not starting on an invariant torus may well travel a long distance in the space of actions. This phenomenon is known as *Arnol'd diffusion*. In fact, Arnol'd has formulated the following conjecture: for "generic" perturbations $H_1(I, \varphi, \varepsilon)$ (in a sense to be defined), there exist a trajectory and times t_0, t_1 such that

$$|I(t_1) - I(t_0)| \geqslant 1. \tag{3.3.51}$$

However, this conjecture has not been proved yet. Nekhoroshev has shown that if H is analytic, then the diffusion time must be very long: If a trajectory satisfying (3.3.51) exists, then

$$t_1 - t_0 \geqslant \mathrm{e}^{1/\varepsilon^\eta}, \qquad \eta > 0. \tag{3.3.52}$$

For certain trajectories, diffusion times admit even larger lower bounds, such as $\mathrm{e}^{\mathrm{e}^{1/\varepsilon}}$. Thus, although systems with three or more degrees of freedom are not necessarily stable for infinite times, in practice they are often stable on a very long time scale if the perturbation is small.

Bibliographical Comments

Hamiltonian systems are discussed in detail in [Ar89]. The averaging theorem is found in the classical textbooks [GH83] and [Wi90], as well as [V96]. The method known as Lie–Deprit series was introduced in [D69], and has been extended to non-Hamiltonian systems [H70].

Kolmogorov's outline of proof appeared in [K54], and Arnol'd's general result in [Ar63]. The first version of Moser's theorem appeared in [Mo62] and was improved by Rüssmann [Rü70] and Herman [H83]. Some comments on Newton's method are found in [Rü72]. Some applications are discussed in [Mo73].

A good review of the properties of twist maps is [Me92]. There have been numerous developments in KAM theory. See for instance [dlL01] for a review and additional references.

Chapter 4

Singular Perturbation Theory

In this last chapter, we will consider so-called *slow–fast systems* of the form

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y), \\ \dot{y} &= g(x,y), \end{aligned} \tag{4.0.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, f and g are of class $\mathcal{C}^r$, $r \geqslant 2$, in some domain $\mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m$, and ε is a small parameter. The variable x is called *fast variable* and y is called *slow variable*, because $\dot{x}/\dot{y}$ can be of order $1/\varepsilon$.

Equation (4.0.1) is an example of a *singularly perturbed* system, because in the limit $\varepsilon \to 0$, if does not reduce to a differential equation of the same type, but to an algebraic–differential system

$$\begin{aligned} 0 &= f(x,y) \\ \dot{y} &= g(x,y). \end{aligned} \tag{4.0.2}$$

Since this limiting system is not an ordinary differential equation, it is not clear how solutions of (4.0.1) might be expanded into powers of ε, for instance. The set of points $(x,y) \in \mathcal{D}$ for which $f(x,y) = 0$ is called the *slow manifold*. If we assume that this manifold can be represented by an equation $x = x^\star(y)$, then the dynamics of (4.0.2) on the slow manifold is governed by the equation

$$\dot{y} = g(x^\star(y), y) =: G(y), \tag{4.0.3}$$

which is easier to solve than the original equation (4.0.1). One of the questions one tries to answer in singular perturbation theory is thus: What is the relation between solutions of the singularly perturbed system (4.0.1) and those of the reduced system (4.0.3)?

There is another way to study the singular limit $\varepsilon \to 0$. With respect to the *fast time* $s = t/\varepsilon$, (4.0.1) can be written as

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}s} &= f(x,y) \\ \frac{\mathrm{d}y}{\mathrm{d}s} &= \varepsilon g(x,y). \end{aligned} \tag{4.0.4}$$

Taking the limit $\varepsilon \to 0$, we obtain the so-called *associated system*

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f(x,y), \qquad\qquad y = \textit{constant}, \tag{4.0.5}$$

in which y plays the rôle of a parameter. The perturbed system in the form (4.0.4) can thus be considered as a modification of the associated system (4.0.5) in which the "parameter" y changes slowly in time. Observe that the slow manifold $x = x^\star(y)$ defines equilibrium points of the associated system.

Results from regular perturbation theory (e.g. Corollary 3.1.7) show that orbits of (4.0.4) will remain close to those of (4.0.5) for s of order 1, i.e., for t of order ε, but not necessarily for larger t. Our aim is to describe the dynamics for times t of order 1.

We will start by giving sufficient conditions under which (4.0.3) and (4.0.5) indeed provide good approximations to different phases of the motion. Then we will discuss some cases in which these conditions are violated, and new phenomena occur.

4.1 Slow Manifolds

In order to give an idea of what to expect in the general case, let us first discuss a solvable example.

Example 4.1.1. Consider the equation

$$\begin{aligned} \varepsilon\dot{x} &= -x + \sin(y) \\ \dot{y} &= 1. \end{aligned} \tag{4.1.1}$$

Then the slow manifold is given by the curve

$$x = x^\star(y) = \sin(y), \tag{4.1.2}$$

and the dynamics reduced to the slow manifold is described by

$$x(t) = \sin(y(t)), \qquad y(t) = y_0 + t. \tag{4.1.3}$$

The associated system for $s = t/\varepsilon$ is

$$\frac{\mathrm{d}x}{\mathrm{d}s} = -x + \sin(y), \qquad y = \textit{constant}, \tag{4.1.4}$$

and admits the solution

$$x(s) = (x_0 - \sin(y))\,\mathrm{e}^{-s} + \sin(y). \tag{4.1.5}$$

Thus all solutions converge to the slow manifold $x = \sin(y)$ as $s \to \infty$.

Let us now compare the approximate solutions to the solution of the original system (4.1.1) which can, in this case, be computed by the method of variation of the constant. It can be written in the form

$$\begin{aligned} x(t) &= \bigl[x_0 - \bar{x}(0,\varepsilon)\bigr]\,\mathrm{e}^{-t/\varepsilon} + \bar{x}(t,\varepsilon), \qquad \bar{x}(t,\varepsilon) = \frac{\sin(y_0+t) - \varepsilon\cos(y_0+t)}{1+\varepsilon^2}, \\ y(t) &= y_0 + t. \end{aligned} \tag{4.1.6}$$

For small t, we have $\bar{x}(t,\varepsilon) = \sin(y_0) + \mathcal{O}(\varepsilon) + \mathcal{O}(t)$, so that

$$x(t) = \bigl[x_0 - \sin(y_0)\bigr]\,\mathrm{e}^{-t/\varepsilon} + \sin(y_0) + \mathcal{O}(\varepsilon) + \mathcal{O}(t), \tag{4.1.7}$$

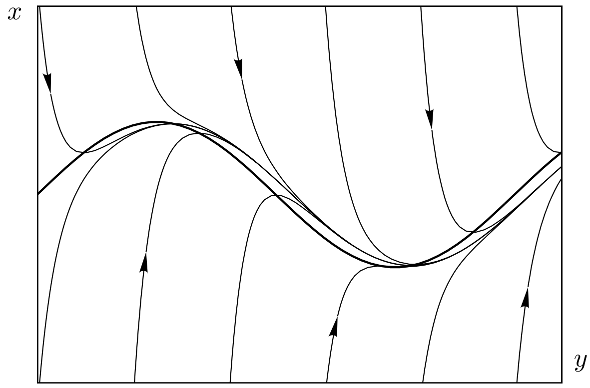

FIGURE 4.1. The heavy curve represents the slow manifold (4.1.2), while the light curves represent solutions of the slow–fast equation (4.1.1) for various initial conditions, and $\varepsilon = 0.07$. The solutions are quickly attracted by a small neighbourhood of order ε of the slow manifold.

which is well approximated by the solution (4.1.5) of the associated system. As time t grows, however, the drift of y makes itself felt and the associated system no longer provides a good approximation.

On the other hand, the factor $\mathrm{e}^{-t/\varepsilon}$ decreases very fast: For $t = k\varepsilon|\log\varepsilon|$, for instance, it is equal to ε^k, and for larger times, it goes to zero faster than any power of ε. The solution (4.1.6) thus becomes very close to

$$x(t) = \frac{\sin(y(t)) - \varepsilon\cos(y(t))}{1+\varepsilon^2}, \qquad y(t) = y_0 + t, \tag{4.1.8}$$

which lies at a distance of order ε from the slow manifold $x = \sin(y)$. Thus the reduced equation (4.1.3) on the slow manifold provides a good approximation of the system (4.1.1) for $t \gg \varepsilon|\log\varepsilon|$.

Tihonov's theorem gives sufficient conditions under which this behaviour holds for general systems: Solutions are attracted, in a time of order $\varepsilon|\log\varepsilon|$, to a small neighbourhood of order ε of a slow manifold.

4.1.1 Tihonov's Theorem

We consider the slow–fast system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{4.1.9}$$

with $f \in \mathcal{C}^r(\mathcal{D},\mathbb{R}^n)$, $g \in \mathcal{C}^r(\mathcal{D},\mathbb{R}^m)$, $r \geqslant 2$. Let us first give conditions under which the slow manifold, defined implicitly by $f(x,y) = 0$, can be represented in the explicit form $x = x^\star(y)$. We introduce the $n \times n$ matrix

$$A_x(y) = \frac{\partial f}{\partial x}(x,y). \tag{4.1.10}$$

Assume that $f(x_0,y_0) = 0$. The implicit function theorem states that if $\det A_{x_0}(y_0) \neq 0$, then there exists a neighbourhood $\mathcal{N}$ of (x_0,y_0) such that all solutions of $f(x,y) = 0$ in $\mathcal{N}$

can be written as $x = x^\star(y)$. Here $x^\star$ is a function of class $\mathcal{C}^r$, defined in a neighbourhood $\mathcal{N}_0$ of y_0 in $\mathbb{R}^m$, and it satisfies

$$\frac{\partial x^\star}{\partial y}(y) = -A_{x^\star(y)}(y)^{-1}\frac{\partial f}{\partial y}(x^\star(y), y). \tag{4.1.11}$$

We will abbreviate $A_{x^\star(y)}(y)$ by $A(y)$. From now on we will always assume that (x, y) belongs to the open set $\mathcal{N}$, and that

1. $\det A(y)$ is bounded away from 0 whenever $(x^\star(y), y) \in \mathcal{N}$,
2. the norms of f, g and all their mixed partial derivatives up to order 2 are bounded uniformly in $\mathcal{N}$.

We now state a result describing how well the slow–fast system (4.1.9) is approximated by its limit when $\varepsilon \to 0$. The first versions of this result were proved independently by Tihonov and Gradšteĭn.

Theorem 4.1.2. *Assume that the eigenvalues $a_j(y)$ of $A(y)$ satisfy*

$$\operatorname{Re} a_j(y) \leqslant -a_0 \qquad \forall y \in \mathcal{N}_0, \quad j = 1, \dots, n \tag{4.1.12}$$

for some constant $a_0 > 0$. Let f and g, as well as their derivatives up to order 2, be uniformly bounded in $\mathcal{N}$. Then there exist constants ε_0, c_0 to c_4, $K_0 > 0$ such that the following properties hold for $0 < \varepsilon < \varepsilon_0$.

1. *Any solution of (4.1.9) with initial condition $(x_0, y_0) \in \mathcal{N}$ such that $\|x_0 - x^\star(y_0)\| < c_0$ satisfies*
$$\|x(t) - x^\star(y(t))\| \leqslant c_1\varepsilon + c_2\|x_0 - x^\star(y_0)\|\,\mathrm{e}^{-K_0 t/\varepsilon} \tag{4.1.13}$$
for all $t \geqslant 0$ such that $(x(s), y(s)) \in \mathcal{N}$ for $0 \leqslant s \leqslant t$.
2. *Let $y^0(t)$ be the solution of the reduced system*
$$\dot y^0 = G(y^0) := g(x^\star(y^0), y^0) \tag{4.1.14}$$
with initial condition $y^0(0) = y_0$. Let K_1 be a Lipschitz constant for G in $\mathcal{N}_0$. Then
$$\|y(t) - y^0(t)\| \leqslant c_3\varepsilon\,\mathrm{e}^{K_1 t} + c_4\|x_0 - x^\star(y_0)\|\,\mathrm{e}^{-K_0 t/\varepsilon} \tag{4.1.15}$$
for all $t \geqslant 0$ such that $(x(s), y(s)) \in \mathcal{N}$ for $0 \leqslant s \leqslant t$.

Roughly speaking, Relation (4.1.13) implies that

$$x(t) = x^\star(y(t)) + \mathcal{O}(\varepsilon) \qquad\qquad \text{for } t \geqslant \frac{1}{K_0}\varepsilon|\log\varepsilon|, \tag{4.1.16}$$

and thus $x(t)$ reaches a neighbourhood of order ε of the slow manifold in a time of order $\varepsilon|\log\varepsilon|$, and stays there as long as this manifold is attracting. If we want to determine $x(t)$ more precisely, we need to approximate $y(t)$ as well. This is done by Relation (4.1.15), which implies that we also have

$$\begin{aligned} y(t) &= y^0(t) + \mathcal{O}(\varepsilon) \\ x(t) &= x^\star(y^0(t)) + \mathcal{O}(\varepsilon) \end{aligned} \qquad\qquad \text{for } \frac{1}{K_0}\varepsilon|\log\varepsilon| \leqslant t \leqslant \frac{1}{K_1}. \tag{4.1.17}$$

Thus if we can solve the simpler reduced equation (4.1.14), then we can compute x and y up to errors of order ε and for times of order 1.

PROOF OF THEOREM 4.1.2. We will give a proof in the case $n = 1$, and comment on larger n at the end of the proof. For $n = 1$, $A(y) = a(y)$ is a scalar and $a(y) \leqslant -a_0$ $\forall y$.

1. **Change of variables:** The deviation of general solutions of (4.1.9) from the slow manifold $x = x^\star(y)$ is characterized by the variable $z = x - x^\star(y)$, which satisfies the equation

$$\varepsilon \dot{z} = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\bigl(x - x^\star(y)\bigr) = f(x^\star(y) + z, y) - \varepsilon \frac{\partial x^\star}{\partial y}(y)\, g(x^\star(y) + z, y).$$

The variables (y, z) belong to $\mathcal{N}' = \{(y, z) \colon (x^\star(y) + z, y) \in \mathcal{N}\}$. We will expand f in Taylor series to second order in z, and write the system in the form

$$\begin{aligned} \varepsilon \dot{z} &= a(y) z + b(y, z) + \varepsilon w(y, z) \\ \dot{y} &= g(x^\star(y) + z, y). \end{aligned}$$

The nonlinear term

$$b(y, z) = f(x^\star(y) + z, y) - a(y) z$$

satisfies, by Taylor's formula and our assumption on the derivatives,

$$\|b(y, z)\| \leqslant M \|z\|^2 \qquad \forall (y, z) \in \mathcal{N}'$$

for some constant $M > 0$. The *drift term* is given by (see (4.1.11))

$$w(y, z) = -\frac{\partial x^\star}{\partial y}(y)\, g(x^\star(y) + z, y) = a(y)^{-1} \frac{\partial f}{\partial y}(x^\star(y), y)\, g(x^\star(y) + z, y).$$

Our assumptions on f, g and a imply the existence of a constant $W > 0$ such that

$$\|w(y, z)\| \leqslant W \qquad \forall (y, z) \in \mathcal{N}'.$$

If w were absent, the slow manifold $z = 0$ would be invariant. The drift term acts like an inertial force, and may push solutions away from the slow manifold.

2. **Proof of** (4.1.13)**:** The above bounds on a, b and w imply that

$$\begin{aligned} \varepsilon \dot{z} &\leqslant -a_0 z + M z^2 + \varepsilon W && \text{if } z \geqslant 0 \\ \varepsilon \dot{z} &\geqslant -a_0 z - M z^2 - \varepsilon W && \text{if } z \leqslant 0 \end{aligned}$$

whenever $(y, z) \in \mathcal{N}$. Thus if $\varphi(t) = |z(t)|$, we have

$$\varepsilon \dot{\varphi}(t) \leqslant -a_0 \varphi(t) + M \varphi(t)^2 + \varepsilon W.$$

The function $\varphi(t)$ is not differentiable at those t for which $\varphi = 0$, but it is left and right differentiable at these points, so that this small inconvenience will not cause any problems. Now we take

$$c_0 = \frac{a_0}{2M}, \qquad c_1 = \frac{2W}{a_0}, \qquad \varepsilon_0 = \frac{a_0}{2Mc_1},$$

so that $\varphi(0) = |z(0)| < \frac{a_0}{2M}$ and $c_1 \varepsilon < \frac{a_0}{2M}$ under the hypotheses of the theorem. We define

$$\tau = \inf\Bigl\{t > 0 \colon \varphi(t) = \frac{a_0}{2M} \text{ or } (y(t), z(t)) \notin \mathcal{N}'\Bigr\}$$

with the convention that $\tau = \infty$ if the set on the right-hand side is empty. By continuity of the solutions, this definition implies that

$$\tau < \infty \qquad \Rightarrow \qquad \varphi(\tau) = \frac{a_0}{2M} \quad \text{or} \quad (y(\tau), z(\tau)) \in \partial\mathcal{N}'.$$

For $0 \leqslant t \leqslant \tau$, we have

$$\varepsilon\dot{\varphi}(t) \leqslant -\frac{a_0}{2}\varphi(t) + \varepsilon W.$$

We cannot apply our version of Gronwall's Lemma directly, because the coefficient of $\varphi(t)$ is negative. We can, however, bound the difference between $\varphi(t)$ and the value we would obtain if this relation were an equality. We define a function $\psi(t)$ by

$$\varphi(t) = \frac{2\varepsilon}{a_0}W + \Bigl(\varphi(0) - \frac{2\varepsilon}{a_0}W + \psi(t)\Bigr)\,\mathrm{e}^{-a_0 t/2\varepsilon}\,.$$

Differentiating with respect to time and using the inequality for $\dot{\varphi}$, we obtain that $\dot{\psi}(t) \leqslant 0$. Since $\psi(0) = 0$, we thus have

$$\varphi(t) \leqslant \frac{2\varepsilon}{a_0}W + \Bigl(\varphi(0) - \frac{2\varepsilon}{a_0}W\Bigr)\,\mathrm{e}^{-a_0 t/2\varepsilon} < \frac{a_0}{2M} \quad \forall t \in [0, \tau].$$

Our definition of c_0, c_1, ε_0 implies that $\varphi(t) < \frac{a_0}{2M}$ for $0 \leqslant t \leqslant \tau$. If $\tau < \infty$, we have $\varphi(\tau) < \frac{a_0}{2M}$ and thus necessarily $(y(\tau), z(\tau)) \in \partial\mathcal{N}'$. In other words, the above upper bound for $\varphi(t)$ holds for all t such that $(y(s), z(s)) \in \mathcal{N}'$ for $s \in [0, t]$. Since $\varphi(t) = \|x(t) - x^\star(y(t))\|$, we have proved (4.1.13).

3. **Proof of** (4.1.15)**:** This part of the proof is similar to proofs in the chapter on regular perturbation theory. We want to compare solutions of the equations

$$\begin{aligned}
\dot{y} &= g(x^\star(y) + z, y) =: G(y) + zR(y, z) \\
\dot{y}^0 &= g(x^\star(y^0), y^0) =: G(y^0).
\end{aligned}$$

Taylor's formula shows that $\|R(y, z)\| \leqslant M'$ for some constant M' uniformly in $\mathcal{N}'$. If $\eta(t) = y(t) - y^0(t)$, we find

$$\dot{\eta} = G(y^0 + \eta) - G(y^0) + zR(y^0 + \eta, z).$$

Integrating from time 0 to t, using K_1 as a Lipschitz constant for G and (4.1.13), we obtain

$$|\eta(t)| \leqslant K_1 \int_0^t |\eta(s)|\,\mathrm{d}s + M'(c_1\varepsilon + c_2\|x_0 - x^\star(y_0)\|\,\mathrm{e}^{-K_0 t/\varepsilon}).$$

Now the bound (4.1.15) follows easily from Gronwall's inequality and a short computation.

4. **The case** $n > 1$**:** In order to reduce the problem to a one-dimensional one, it is possible to use a so-called *Liapunov function* $V : \mathbb{R}^n \to \mathbb{R}$, which decreases along orbits of the associated system. A possibility is to use a quadratic form

$$V(y, z) = z \cdot Q(y)z,$$

for some suitable symmetric positive definite matrix $Q(y)$, depending on $A(y)$. Then the function $\varphi(t) = V(y(t), z(t))^{1/2}$ obeys a similar equation as in the case $n = 1$. □

4.1.2 Iterations and Asymptotic Series

In the proof of Theorem 4.1.2, we have used the fact that the original system

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y) \\ \dot{y} &= g(x,y) \end{aligned} \tag{4.1.18}$$

is transformed, by the translation $x = x^\star(y) + z$, into the system

$$\begin{aligned} \varepsilon\dot{z} &= f(x^\star(y)+z,y) - \varepsilon\frac{\partial x^\star}{\partial y}(y)g(x^\star(y)+z,y) \\ \dot{y} &= g(x^\star(y)+z,y). \end{aligned} \tag{4.1.19}$$

We can rewrite this system in the following way:

$$\begin{aligned} \varepsilon\dot{z} &= A(y)z + b_0(y,z,\varepsilon) + \varepsilon w_0(y) \\ \dot{y} &= g_0(y,z), \end{aligned} \tag{4.1.20}$$

where $A(y) = \frac{\partial f}{\partial x}(x^\star(y),y)$, and $w_0(y) = -\frac{\partial}{\partial y}x^\star(y)g(x^\star(y),y)$ contains the terms which do not vanish at $z=0$. The term b_0 contains all remainders of the Taylor expansion of f (at order 2) and g (at order 1), and is bounded in norm by a constant times $\|z\|^2 + \varepsilon\|z\|$.

We know that $z(t)$ is asymptotically of order ε. The reason why z does not go to zero is that the drift term $\varepsilon w_0(y)$ may push z away from zero. If we manage to increase the order of the drift term, then z may become smaller and we will have obtained a better approximation of the solution of (4.1.18).

We could use the fact that (4.1.20) is again a slow–fast system, where $\dot{z}$ vanishes on a slow manifold $z = z^\star(y)$. Since $z^\star(y)$ is in general difficult to compute, we will only use an approximation of it. Consider the change of variables

$$z = z_1 + \varepsilon u_1(y), \qquad u_1(y) = -A(y)^{-1}w_0(y). \tag{4.1.21}$$

Substituting into (4.1.20), we get

$$\varepsilon\dot{z}_1 = A(y)z_1 + b_0(y, z_1+\varepsilon u_1(y)) - \varepsilon^2\frac{\partial u_1}{\partial y}g_0(y, z_1+\varepsilon u_1(y)). \tag{4.1.22}$$

We can again extract the terms which vanish for $z=0$, which are of order ε^2. The system can thus be written as

$$\begin{aligned} \varepsilon\dot{z}_1 &= A(y)z_1 + b_1(y,z_1,\varepsilon) + \varepsilon^2 w_1(y,\varepsilon) \\ \dot{y} &= g_1(y,z_1,\varepsilon), \end{aligned} \tag{4.1.23}$$

where w_1 denotes the right-hand side of (4.1.22) for $z_1 = 0$ and Taylor's formula shows again that $b_1(y,z,\varepsilon)$ is of order $\|z\|^2 + \varepsilon\|z\|$. As long as the system is sufficiently differentiable, we can repeat this procedure, increasing the order in ε of the drift term by successive changes of variables. Thus there is a transformation

$$x = x^\star(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) + \dots + \varepsilon^r u_r(y) + z_r \tag{4.1.24}$$

which yields a system of the form

$$\begin{aligned} \varepsilon\dot{z}_r &= A(y)z_r + b_r(y,z_r,\varepsilon) + \varepsilon^{r+1} w_r(y,\varepsilon) \\ \dot{y} &= g_r(y,z_r,\varepsilon). \end{aligned} \tag{4.1.25}$$

Proceeding as in the proof of Tihonov's theorem, one can show that $z_r(t) = \mathcal{O}(\varepsilon^{r+1})$ after a time of order $\varepsilon|\log\varepsilon|$. This implies that after this time, solutions of (4.1.18) satisfy

$$\begin{aligned} x(t) &= x^\star(y(t)) + \varepsilon u_1(y(t)) + \dots + \varepsilon^r u_r(y(t)) + \mathcal{O}(\varepsilon^{r+1}) \\ \dot y &= g(x^\star(y) + \varepsilon u_1(y) + \dots + \varepsilon^r u_r(y), y) + \mathcal{O}(\varepsilon^{r+1}) \end{aligned} \tag{4.1.26}$$

for times of order 1.

One might wonder whether these iterations may be pushed all the way to $r = \infty$ if the system is analytic, in such a way that the remainder disappears. The answer in negative in general, because the amplitude of the drift term w_r tends to grow with r. We will illustrate this phenomenon by a very simple example.

Example 4.1.3. Consider the equation

$$\begin{aligned} \varepsilon\dot x &= -x + h(y) \\ \dot y &= 1, \end{aligned} \tag{4.1.27}$$

where we will assume that y is analytic in the strip $|\operatorname{Im} y| \leqslant R$. This system can be solved exactly:

$$x(t) = x_0\,\mathrm{e}^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t \mathrm{e}^{-(t-s)/\varepsilon} h(y_0+s)\,\mathrm{d}s, \qquad y(t) = y_0 + t. \tag{4.1.28}$$

Thus if $|h(y)|$ is uniformly bounded by M, we have

$$|x(t)| \leqslant |x_0|\,\mathrm{e}^{-t/\varepsilon} + \frac{M}{\varepsilon}\int_0^t \mathrm{e}^{-(t-s)/\varepsilon}\,\mathrm{d}s = |x_0|\,\mathrm{e}^{-t/\varepsilon} + M(1-\mathrm{e}^{-t/\varepsilon}). \tag{4.1.29}$$

Thus there is no question that $x(t)$ exists and is uniformly bounded for all $t \geqslant 0$, by a constant independent of ε.

Now let us pretend that we do not know the exact solution (4.1.28). The most naive way to solve (4.1.27) is to look for a series of the form

$$x = x^0(y) + \varepsilon x^1(y) + \varepsilon^2 x^2(y) + \dots \tag{4.1.30}$$

It is unlikely that all solutions admit such a series representation, but if we obtain a particular solution of (4.1.27) which has this form, we can then determine the general solution. Substituting the series (4.1.30) into (4.1.27), and solving order by order, we obtain

$$\begin{aligned} 0 &= -x^0(y) + h(y) &&\Rightarrow && x^0(y) = h(y) \\ (x^0)'(y) &= -x^1(y) &&\Rightarrow && x^1(y) = -h'(y) \\ (x^1)'(y) &= -x^2(y) &&\Rightarrow && x^2(y) = h''(y) \end{aligned} \tag{4.1.31}$$

and so on. Thus x admits a formal series representation of the form

$$x(t) = h(y(t)) - \varepsilon h'(y(t)) + \dots + (-\varepsilon)^k h^{(k)}(y(t)) + \cdots \tag{4.1.32}$$

Can this series converge? In certain cases yes, as shows the particular case $h(y) = \sin y$ that we investigated in Example 4.1.1. In that case, the function $\bar x(t,\varepsilon)$ admits an expansion in ε which converges for $|\varepsilon| < 1$. In general, however, the derivatives $h^{(k)}(y)$ may grow with

k. Cauchy's formula tells us that if h is analytic and bounded by M in a disc of radius R in the complex plane around y, then

$$h^{(k)}(y) = \frac{k!}{2\pi\,\mathrm{i}} \int_{|z-y|=R} \frac{h(z)}{(z-y)^{k+1}}\,\mathrm{d}z \qquad \Rightarrow \qquad |h^{(k)}(y)| \leqslant MR^{-k}k! \tag{4.1.33}$$

Thus in principle, $|h^{(k)}(y)|$ may grow like $k!$, which is too fast for the formal series (4.1.32) to converge.

A more subtle method consists in trying to simplify (4.1.27) by successive changes of variables. The translation $x = h(y) + z$ yields the equation

$$\varepsilon\dot{z} = -z - \varepsilon h'(y). \tag{4.1.34}$$

The drift term $-\varepsilon h'(y)$ can be further decreased by the change of variables $z = -\varepsilon h'(y) + z_1$, and so on. In fact, the transformation

$$x = \sum_{j=0}^{k} (-\varepsilon)^j h^{(j)}(y) + z_k \tag{4.1.35}$$

results in the equation

$$\varepsilon\dot{z}_k = -z_k + (-\varepsilon)^{k+1} h^{(k+1)}(y). \tag{4.1.36}$$

The solution is given by

$$z_k(t) = z_k(0)\,\mathrm{e}^{-t/\varepsilon} - (-\varepsilon)^k \int_0^t \mathrm{e}^{-(t-s)/\varepsilon}\, h^{(k+1)}(y_0 + s)\,\mathrm{d}s. \tag{4.1.37}$$

Of course, the relations (4.1.35) and (4.1.37) can be obtained directly from the exact solution (4.1.28) by k successive integrations by parts. The iterative method, however, also works in cases in which we do not know an exact solution. In that case, the remainder $z_k(t)$ can be bounded as in the proof of Tihonov's theorem.

We are looking for a particular solution admitting an expansion of the form (4.1.32). If we choose $x(0)$ in such a way that $z_k(0) = 0$, we obtain for each $k \in \mathbb{Z}$ a particular solution satisfying

$$x(t,\varepsilon) = \sum_{j=0}^{k} (-\varepsilon)^j h^{(j)}(y(t)) + z_k(t,\varepsilon) \tag{4.1.38}$$

$$|z_k(t,\varepsilon)| \leqslant \varepsilon^{k+1} MR^{-(k+1)}(k+1)! \tag{4.1.39}$$

where M, R are independent of k. The first k terms of this expansion agree with the formal solution (4.1.32). In general, an expression of the form

$$x(t,\varepsilon) = \sum_{j=0}^{k} \varepsilon^j u_j(t) + \varepsilon^{k+1} r_k(t,\varepsilon) \qquad \forall k \in \mathbb{Z}, \tag{4.1.40}$$

where r_k is bounded for all k, but the series does not necessarily converge, is called an *asymptotic series*. If the remainder grows like in (4.1.39), this asymptotic series is called of type *Gevrey-1*. This situation is common in singularly perturbed systems: bounded solutions exist, but they do not admit convergent series in ε (this differs from the situation we encountered in Chapter 3, where the existence of a bounded solution is not always guaranteed).

Remark 4.1.4. We can determine at which order k to stop the asymptotic expansion (4.1.38) in such a way that the remainder $\varepsilon^{k+1}|z_k|$ is a small as possible. This order will be approximately the one for which $k!\varepsilon^k$ is minimal. Recall Stirling's formula

$$k! \simeq k^k \,\mathrm{e}^{-k} \,. \tag{4.1.41}$$

It is convenient to optimize the logarithm of $k!\varepsilon^k$. We have

$$\frac{\mathrm{d}}{\mathrm{d}k} \log\bigl(k^k \,\mathrm{e}^{-k}\, \varepsilon^k\bigr) = \frac{\mathrm{d}}{\mathrm{d}k}\bigl(k\log k - k + k\log\varepsilon\bigr) = \log k + \log\varepsilon, \tag{4.1.42}$$

which vanishes for $\log k = -\log\varepsilon$, i.e., $k = 1/\varepsilon$ (of course we need to take an integer k but this hardly makes a difference for small ε). For this k, we have

$$k!\varepsilon^k \simeq \Bigl(\frac{1}{\varepsilon}\Bigr)^{1/\varepsilon} \mathrm{e}^{-1/\varepsilon}\,\varepsilon^{1/\varepsilon} = \mathrm{e}^{-1/\varepsilon}\,. \tag{4.1.43}$$

We conclude that the optimal truncation of the series is for $k = \mathcal{O}(1/\varepsilon)$, which yields a remainder of order $\mathrm{e}^{-1/\varepsilon}$. This function goes to zero faster than any power of ε when $\varepsilon \to 0$, and thus even though we cannot compute the solution exactly in general, we can compute it to a relatively high degree of precision.

For general slow–fast systems one can show that if f and g are analytic in some open complex domain, then the iterative scheme leading to (4.1.25) can also be truncated in such a way that the remainder is exponentially small in ε. We will not prove this result here, but only mention that it heavily relies on Cauchy's formula.

Remark 4.1.5. There is another way which may help to understand why the asymptotic series (4.1.32) does not converge for general h. Assume that h is periodic in y, and write its Fourier series in the form

$$h(y) = \sum_{k\in\mathbb{Z}} h_k \,\mathrm{e}^{\mathrm{i}\,ky}, \qquad\qquad h_k = \frac{1}{2\pi}\int_0^{2\pi} h(y)\,\mathrm{e}^{-\mathrm{i}\,ky}\,\mathrm{d}y. \tag{4.1.44}$$

The formal series (4.1.32) will also be periodic in y, so that we may look for a solution of (4.1.27) of the form

$$x(y) = \sum_{k\in\mathbb{Z}} x_k \,\mathrm{e}^{\mathrm{i}\,ky}\,. \tag{4.1.45}$$

Substituting in the equation for x, we obtain

$$\varepsilon\,\mathrm{i}\,kx_k = -x_k + h_k \qquad \forall k \in \mathbb{Z}\,, \tag{4.1.46}$$

and thus the periodic solution can be written as

$$x(y) = \sum_{k\in\mathbb{Z}} \frac{h_k}{1+\varepsilon\,\mathrm{i}\,k}\,\mathrm{e}^{\mathrm{i}\,ky}\,. \tag{4.1.47}$$

This Fourier series *does* converge in a neighbourhood of the real axis because the h_k decrease exponentially fast in $|k|$ (to see this, shift the integration path of h_k in (4.1.44) by an imaginary distance). Thus the equation admits indeed a bounded periodic solution. However, in the plane of complex ε, the function (4.1.47) has poles at every $\varepsilon = -\,\mathrm{i}\,/k$ for which $h_k \neq 0$. In the case $h(y) = \sin y$, there are only two poles at $\varepsilon = \pm\,\mathrm{i}$, so that the series converges for $|\varepsilon| < 1$. If $h(y)$ has nonvanishing Fourier components with arbitrarily large k, however, the poles accumulate at $\varepsilon = 0$, and the radius of convergence of the series in ε is equal to zero.

4.2 Dynamic Bifurcations

We examine now what happens to solutions of the slow–fast system

$$\begin{aligned}\varepsilon\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned} \tag{4.2.1}$$

in cases where the assumptions of Tihonov's theorem are violated. If the slow manifold is represented by $x = x^\star(y)$, the main assumption of Tihonov's theorem is that all eigenvalues of the matrix

$$A(y) = \frac{\partial f}{\partial x}(x^\star(y), y) \tag{4.2.2}$$

have a strictly negative real part. This is equivalent to saying that $x^\star(y)$ is an asymptotically stable equilibrium point of the associated system

$$\frac{\mathrm{d}x}{\mathrm{d}s} = f(x,y), \qquad\qquad y = \mathit{constant}. \tag{4.2.3}$$

A *dynamic bifurcation* occurs if the slow motion of y causes some eigenvalues of $A(y)$ to cross the imaginary axis. Then there may be a bifurcation in the associated system (4.2.3), and the main assumption of Tihonov's theorem is no longer satisfied.

The two most generic cases (codimension 1 bifurcations of equilibria) are

- Hopf bifurcation: a pair of complex conjugate eigenvalues of A cross the imaginary axis; the slow manifold $x^\star(y)$ continues to exist, but changes its stability.
- Saddle–node bifurcation: an eigenvalue of $A(y)$ vanishes, and the slow manifold $x^\star(y)$ ceases to exist (in fact, it has a fold).

We will end this overview by discussing a few relatively simple examples of these two dynamic bifurcations.

4.2.1 Hopf Bifurcation

Example 4.2.1. Consider the slow–fast system

$$\begin{aligned}\varepsilon\dot{x}_1 &= yx_1 - x_2 - x_1(x_1^2 + x_2^2)\\ \varepsilon\dot{x}_2 &= x_1 + yx_2 - x_2(x_1^2 + x_2^2)\\ \dot{y} &= 1.\end{aligned} \tag{4.2.4}$$

The associated system is

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}s} &= yx_1 - x_2 - x_1(x_1^2 + x_2^2)\\ \frac{\mathrm{d}x_2}{\mathrm{d}s} &= x_1 + yx_2 - x_2(x_1^2 + x_2^2)\\ y &= \mathit{constant}.\end{aligned} \tag{4.2.5}$$

The slow manifold is given by $x^\star(y) \equiv (0,0)$, and the matrix

$$A(y) = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix} \tag{4.2.6}$$

has eigenvalues $a_{\pm}(y) = y \pm \mathrm{i}$. Thus there is a Hopf bifurcation at $y = 0$. In fact the complex variable $z = x_1 + \mathrm{i}\, x_2$ satisfies

$$\frac{\mathrm{d}z}{\mathrm{d}s} = (y + \mathrm{i})z - |z|^2 z, \tag{4.2.7}$$

which becomes, in polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$,

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}s} &= yr - r^3 \\ \frac{\mathrm{d}\varphi}{\mathrm{d}s} &= 1. \end{aligned} \tag{4.2.8}$$

If $y \leqslant 0$, all solutions spiral to the origin $r = 0$, while for $y > 0$, the origin is unstable, and all solutions not starting in $r = 0$ are attracted by the periodic orbit $r = \sqrt{y}$.

Let us now return to the slow–fast system (4.2.4). We know, by Tihonov's theorem, that solutions starting not too far from $x = 0$ at some negative y will reach a neighbourhood of order ε of the origin after a time of order $\varepsilon|\log \varepsilon|$. What happens as y approaches zero? Since the origin becomes unstable, intuition would probably tell us that the solution will leave the neighbourhood of the origin as soon as y becomes positive, and approach the periodic orbit.

This assumption can be checked by computing the solutions explicitly. In the complex variable z, the slow–fast system (4.2.4) becomes

$$\varepsilon\dot{z} = (y + \mathrm{i})z - |z|^2 z, \qquad \dot{y} = 1, \tag{4.2.9}$$

and in polar coordinates,

$$\begin{aligned} \varepsilon\dot{r} &= yr - r^3 \\ \varepsilon\dot{\varphi} &= 1 \\ \dot{y} &= 1. \end{aligned} \tag{4.2.10}$$

The solution with initial condition $(r, \varphi, y)(0) = (r_0, \varphi_0, y_0)$ can be written as

$$\begin{aligned} r(t) &= r_0 c(t)\, \mathrm{e}^{\alpha(t)/\varepsilon} \\ \varphi(t) &= \varphi_0 + t/\varepsilon \\ y(t) &= y_0 + t, \end{aligned} \tag{4.2.11}$$

where

$$\begin{aligned} \alpha(t) &= \int_0^t y(s)\,\mathrm{d}s = y_0 t + \frac{1}{2}t^2 \\ c(t) &= \Bigl(1 + \frac{2r_0^2}{\varepsilon}\int_0^t \mathrm{e}^{\alpha(s)/\varepsilon}\,\mathrm{d}s\Bigr)^{-1/2}. \end{aligned} \tag{4.2.12}$$

Assume that we start with $y_0 < 0$. Then the function $\alpha(t)$ is negative for $0 < t < -2y_0$. For these t, the term $\mathrm{e}^{\alpha(t)/\varepsilon}$ is small, while $c(t) \leqslant 1$. This implies that $r(t)$ is very small up to times t close to $-2y_0$. However, the bifurcation already happens at time $t = -y_0$.

For $t > -2y_0$, the term $\mathrm{e}^{\alpha(t)/\varepsilon}$ becomes large, but is counteracted by $c(t)$. In fact, it is not hard to show that $r(t)$ approaches $\sqrt{y(t)}$ as $t \to \infty$, which means that the solution (4.2.11) approaches the periodic orbit of the associated system.

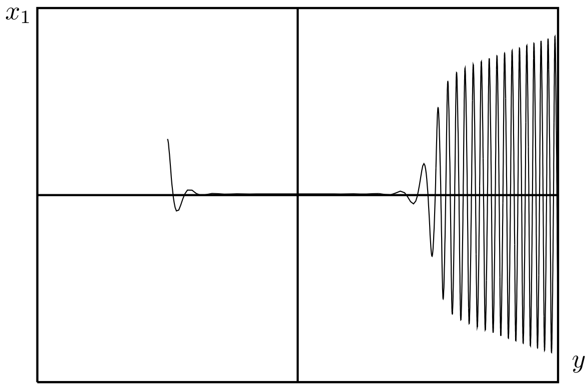

FIGURE 4.2. A solution of Equation 4.2.4 starting with a negative y_0. The Hopf bifurcation occurs at $y = 0$, but the solution approaches the limit cycle at $r = \sqrt{y}$ only after $y = -y_0$. This is the phenomenon of *bifurcation delay.*

In summary, the solution starting at a distance r_0 (independent of ε) from the origin with $y_0 < 0$ is quickly attracted by the origin (in accordance with Tihonov's theorem), but stays there until $y = -y_0$, although the actual bifurcation takes place at $y = 0$. Only after $y = -y_0$ will the solution approach the stable periodic orbit of the associated system (Fig. 4.2). This phenomenon is called *bifurcation delay.*

Example 4.2.2. A particularity of the previous example is that the slow manifold does not depend on y, so that there is no drift term pushing solutions away from it. One might expect that the bifurcation delay is destroyed as soon as $x^\star(y)$ depends on y. Let us consider the following modification of (4.2.4):

$$\begin{aligned} \varepsilon\dot{x}_1 &= y(x_1+y) - x_2 - (x_1+y)((x_1+y)^2 + x_2^2) \\ \varepsilon\dot{x}_2 &= (x_1+y) + yx_2 - x_2((x_1+y)^2 + x_2^2) \\ \dot{y} &= 1. \end{aligned} \tag{4.2.13}$$

The slow manifold is given by $x^\star(y) = (-y, 0)$. The associated system is the same as in Example 4.2.1, up to a translation of $x^\star(y)$. There is still a Hopf bifurcation at $y = 0$.

The variable $z = (x_1 + y) + \mathrm{i}\, x_2$ now obeys the equation

$$\varepsilon\dot{z} = (y+\mathrm{i})z - |z|^2 z + \varepsilon, \qquad \dot{y} = 1. \tag{4.2.14}$$

The additional drift term stems from the y-dependence of the slow manifold.

In naive perturbation theory, one would look for a particular solution of the form

$$z = z^0(y) + \varepsilon z^1(y) + \varepsilon^2 z^2(y) + \cdots \tag{4.2.15}$$

Inserting into (4.2.14) and solving order by order, we get

$$\begin{aligned} 0 &= (y+\mathrm{i})z^0 - |z^0|^2 z^0 & &\Rightarrow & z^0(y) &= 0 \\ \dot{z}^0 &= (y+\mathrm{i})z^1 - \big[(z^0)^2\bar{z}^1 + 2|z^0|^2 z^1\big] + 1 & &\Rightarrow & z^1(y) &= -\frac{1}{y+\mathrm{i}} \\ &\dots \end{aligned} \tag{4.2.16}$$

Thus Equation (4.2.14) seems to admit a particular solution

$$z_{\mathrm{p}}(t) = -\frac{\varepsilon}{y(t)+\mathrm{i}} + \mathcal{O}(\varepsilon^2). \tag{4.2.17}$$

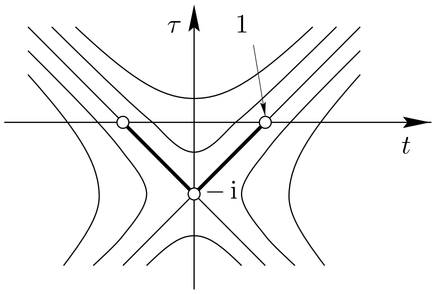

FIGURE 4.3. Level lines of the function $\operatorname{Re}\alpha(t + \mathrm{i}\,\tau)$. The function is small for large $|\tau|$ and large for large $|t|$. The integral $\Psi(t, t_0)$ is small only if t_0 and t can be connected by a path on which $\operatorname{Re}\alpha$ is non-increasing.

To obtain the general solution, we set $z = z_{\mathrm{p}} + \zeta$, substitute in (4.2.14), and use the fact that z_{p} is a particular solution. The result is an equation for ζ of the form

$$\varepsilon\dot{\zeta} = (y + \mathrm{i})\zeta + \mathcal{O}(\varepsilon^2|\zeta|) + \mathcal{O}(\varepsilon|\zeta|^2) + \mathcal{O}(|\zeta|^3). \tag{4.2.18}$$

The drift term has disappeared in the transformation, so that the right-hand side of (4.2.18) vanishes for $\zeta = 0$. Although this equation cannot be solved exactly, it should behave as in Example 4.2.1. To show this, let us assume that the system is started at time $t = t_0 = y_0$ in such a way that $y(0) = 0$. We use the Ansatz

$$\zeta(t) = \zeta_0\,\mathrm{e}^{\alpha(t,t_0)/\varepsilon}\,c(t), \qquad \alpha(t, t_0) = \mathrm{i}(t - t_0) + \frac{1}{2}(t^2 - t_0^2). \tag{4.2.19}$$

If the last three terms in (4.2.18) were absent, the solution would be given by (4.2.19) with $c(t) \equiv 1$. Inserting (4.2.19) into (4.2.18), we obtain a differential equation for c, which can be used to show that c remains of order 1 as long as $\operatorname{Re}\alpha(t, t_0) < 0$. Since $\operatorname{Re}\alpha(t, t_0) < 0$ for $t_0 < t < -t_0$, although the bifurcation happens at $t = 0$, we conclude that the bifurcation is delayed as in the previous example.

There is, however, a major problem with this procedure. We already know that the asymptotic series (4.2.15) is unlikely to converge, but this does not necessarily imply that (4.2.14) does not admit a solution of order ε. In this case, however, (4.2.14) *does not admit a solution of order ε for all times.* To see this, let us write the general solution implicitly as

$$z(t) = z_0\,\mathrm{e}^{\alpha(t,t_0)/\varepsilon} + \frac{1}{\varepsilon}\int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon}\big[\varepsilon - |z(s)|^2 z(s)\big]\,\mathrm{d}s. \tag{4.2.20}$$

We can try to solve this equation by iterations. Assume that $z_0 = \mathcal{O}(\varepsilon)$. As long as z remains of order ε, we have

$$z(t) = z_0\,\mathrm{e}^{\alpha(t,t_0)/\varepsilon} + \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon}\big[1 + \mathcal{O}(\varepsilon^2)\big]\,\mathrm{d}s. \tag{4.2.21}$$

The behaviour of the solution is thus essentially contained in the integral

$$\Psi(t, t_0) = \int_{t_0}^{t} \mathrm{e}^{\alpha(t,s)/\varepsilon}\,\mathrm{d}s. \tag{4.2.22}$$

As long as this integral remains of order ε, $z(t)$ is of order ε. If it becomes large, however, z becomes large as well (though this has to be checked independently). Ψ can be evaluated by deformation of the integration path into the complex plane. The function $\alpha(t) := \alpha(t,0)$ can be continued to the complex plane, and satisfies

$$\operatorname{Re}\alpha(t+\mathrm{i}\,\tau) = -\tau + \frac{1}{2}\big[t^2-\tau^2\big] = \frac{1}{2}\big[t^2-(\tau+1)^2+1\big]. \tag{4.2.23}$$

The level lines of $\operatorname{Re}\alpha$ are hyperbolas centred at $t+\mathrm{i}\,\tau = -\,\mathrm{i}$. The integral (4.2.22) is small if we manage to find a path of integration connecting t_0 to t on which $\operatorname{Re}\alpha(t,s) = \operatorname{Re}\alpha(t) - \operatorname{Re}\alpha(s) \leqslant 0$ for all s. This is only possible if $\operatorname{Re}\alpha(t) \leqslant \operatorname{Re}\alpha(t_0)$, but this condition is not sufficient. In fact (see Fig. 4.3),

- if $t_0 \leqslant -1$, such a path exists for all $t \leqslant 1$;
- if $-1 \leqslant t \leqslant 0$, such a path exists for all $t \leqslant -t_0$;
- if $t_0 \geqslant 0$, no such path exists for $t \geqslant t_0$.

We conclude that if $t_0 \leqslant 0$, then $\Psi(t,t_0)$ remains small for all times

$$t \leqslant \min\{-t_0, 1\}. \tag{4.2.24}$$

As a consequence, $z(t)$ also remains small up to this time. The term $-t_0$ is natural, it is the same as in Example 4.2.1. However, the term 1, called *maximal delay* or *buffer point*, is new. It is an effect of the drift term, and cannot be found by naive perturbation theory, because it is, so to speak, hidden in exponentially small terms. This example shows that one should be extremely cautious with asymptotic expansions near dynamic bifurcations.

4.2.2 Saddle–Node Bifurcation

Example 4.2.3. Let us finally return to Van der Pol's equation, written in the form

$$\begin{aligned} \varepsilon\dot{x} &= y + x - \frac{1}{3}x^3 \\ \dot{y} &= -x. \end{aligned} \tag{4.2.25}$$

The slow manifold is given implicitly by the equation

$$y = \frac{1}{3}x^3 - x. \tag{4.2.26}$$

This equation has three solutions for x if $|y| < 2/3$ and one if $|y| > 2/3$. The associated system

$$\frac{\mathrm{d}x}{\mathrm{d}s} = x - \frac{1}{3}x^3 + y, \qquad y = \textit{constant} \tag{4.2.27}$$

has three equilibria (two stable, one unstable) for $|y| < 2/3$ and one (stable) equilibrium for $|y| > 2/3$. The points $(1,-2/3)$ and $(-1,2/3)$ are saddle–node bifurcation points.

Let us now study the dynamics near $(1,-2/3)$. The change of variables

$$\begin{aligned} x &= 1+\xi \\ y &= -\frac{2}{3} - \eta \end{aligned} \tag{4.2.28}$$

transforms (4.2.25) into the system

$$\begin{aligned} \varepsilon\dot{\xi} &= -\eta - \xi^2 - \frac{1}{3}\xi^3 \\ \dot{\eta} &= 1 + \xi. \end{aligned} \tag{4.2.29}$$

Assume that we start near the piece of slow manifold given by

$$\xi = \xi^\star(\eta) = \sqrt{-\eta} + \mathcal{O}(\eta), \qquad \eta \leqslant 0. \tag{4.2.30}$$

Tihonov's theorem states that for negative η bounded away from zero, ξ quickly reaches a neighbourhood of order ε of $\xi^\star(\eta)$. The transformation $\xi = \xi^\star(\eta) + \xi_1$ yields the new system

$$\begin{aligned} \varepsilon\dot{\xi}_1 &= -\bigl[2\xi^\star(\eta) + (\xi^\star(\eta))^2\bigr]\xi_1 - \bigl[1 + \xi^\star(\eta)\bigr]\xi_1^2 - \frac{1}{3}\xi_1^3 - \varepsilon\frac{\mathrm{d}\xi^\star(\eta)}{\mathrm{d}\eta}(1 + \xi^\star(\eta) + \xi_1) \\ \dot{\eta} &= 1 + \xi^\star(\eta) + \xi_1. \end{aligned} \tag{4.2.31}$$

The order of the drift term can be further decreased by a translation

$$\xi_1 = \xi_2 + \varepsilon u_1(\eta), \qquad u_1(\eta) = -\frac{\mathrm{d}\xi^\star(\eta)}{\mathrm{d}\eta}\frac{1 + \xi^\star(\eta)}{2\xi^\star(\eta) + (\xi^\star(\eta))^2}. \tag{4.2.32}$$

Continuing in this way, we construct an asymptotic series

$$\xi = \xi^\star(\eta) + \varepsilon u_1(\eta) + \varepsilon^2 u_2(\eta) + \cdots \tag{4.2.33}$$

Note, however, that as η approaches 0, we have

$$\xi^\star(\eta) \sim \sqrt{-\eta}, \qquad\qquad u_1(\eta) \sim \frac{1}{\eta} \tag{4.2.34}$$

and further computations will show that $u_2(\eta) \sim |\eta|^{-5/2}$. Thus the successive terms of the asymptotic series decrease in amplitude only if

$$\frac{\varepsilon}{|\eta|} < \sqrt{|\eta|} \qquad \Rightarrow \qquad |\eta| > \varepsilon^{2/3}. \tag{4.2.35}$$

One says that the asymptotic series becomes *disordered* at $\eta = -\varepsilon^{2/3}$. Following the proof of Tihonov's theorem, one can indeed show that ξ behaves like the first terms of the asymptotic series for times smaller than $-\varepsilon^{2/3}$. For larger η, the system has to be described by other means. Since ξ is of order $\varepsilon^{1/3}$ when $\eta = -\varepsilon^{2/3}$, we introduce the scaling

$$\begin{aligned} \xi &= \varepsilon^{1/3} u \\ \eta &= \varepsilon^{2/3} v, \end{aligned} \tag{4.2.36}$$

which transforms (4.2.29) into

$$\begin{aligned} \varepsilon^{4/3}\dot{u} &= -\varepsilon^{2/3} v - \varepsilon^{2/3} u^2 - \frac{1}{3}\varepsilon u \\ \varepsilon^{2/3}\dot{v} &= 1 + \varepsilon^{1/3} u, \end{aligned} \tag{4.2.37}$$

so that

$$\frac{\mathrm{d}u}{\mathrm{d}v} = -u^2 - v + \mathcal{O}(\varepsilon^{1/3}). \tag{4.2.38}$$

This is a perturbation of order $\varepsilon^{1/3}$ of a solvable Riccati equation, that can be used to describe the motion near the bifurcation point. We will not discuss this equation in detail,

(a) (b)

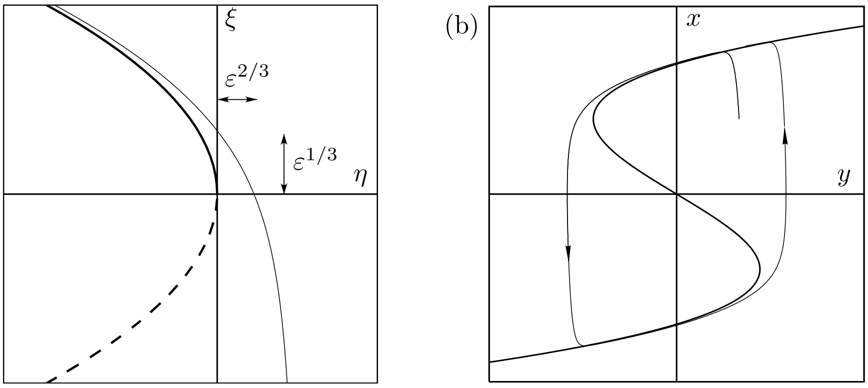

FIGURE 4.4. (a) Behaviour of a solution of (4.2.29) near the saddle–node bifurcation point. The trajectory jumps after a delay of order $\varepsilon^{2/3}$. (b) This jump results in relaxation oscillations for the global behaviour of van der Pol's equation.

but only mention that u can be shown to reach -1 after a "time" v of order 1. This implies that ξ reaches $-\varepsilon^{1/3}$ for η of order $\varepsilon^{2/3}$ (Fig. 4.4a).

Once the bifurcation point has been passed, the solution reaches another branch of the slow manifold, which it follows until the next bifurcation point. The result is an asymptotically periodic motion, with alternating slow and fast phases. In the limit $\varepsilon \to 0$, the motion follows branches of the slow manifold and lines $y = \textit{constant}$. When $\varepsilon > 0$, this cycle is enlarged by an amount of order $\varepsilon^{1/3}$ in the x-direction and of order $\varepsilon^{2/3}$ in the y-direction (Fig. 4.4b).

Bibliographical Comments

Various approaches to singular perturbation theory have been developed, based on asymptotic expansions [MKKR94, VBK95], invariant manifolds [F79, J95] and nonstandard analysis [Be91]. More information on asymptotic series can be found in [Wa65].

Tihonov's theorem was originally proved in [Ti52] and [Gr53], and has been generalized to periodic orbits [PR60].

The phenomenon of delayed Hopf bifurcation has been analysed in detail by Neishtadt [Ne87, Ne88]. The saddle–node bifurcation was first considered in [Po57], and later in [Ha79, MKKR94]. See also [LS77] and [Be91] for other dynamic bifurcations.

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